GRAMMAR SCHOOL ALGEBRA

A. W. POTTER

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BY

A. W. POTTER

FORMERLY SUPERINTENDENT OF SCHOOLS, WILKESBARRE, AND
INSTRUCTOR IN MATHEMATICS, UNIVERSITY OF MICHIGAN

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PREFACE

The growing demand for algebra in the grammar schools has led to the preparation of this brief book; — an algebra which shall, in a year's school work, open up the subject in a simple and comprehensive manner, arouse interest, and lay the foundation for more effective work in the high school; and which shall, at the same time, place in the hands of pupils who leave school at the end of the grammar course a key that will unlock many of the mathematical intricacies they will meet.

Although arithmetic is but the special application of algebra, the former, dealing as it does with the concrete and particular, is developed first, and the pupil has had seven or eight years of number work before he is introduced to the general subject of "literal arithmetic." In the ordinary method of treating algebra the student has been taught what, to him, is new reasoning with new nomenclature. It is the plan in this book to avoid this gap, and to tie the development of the principles of algebraic relations of quantities on to the special parallel cases with which the pupil is familiar in numbers.

The order of treatment of the subject has been suggested by the new course of study of the New York City schools, as it seems to be the most rational and logical.

This plan provides for the simple introduction of the various subjects in the first half of the year, with a review and an extended development in the second half.

The plan may be seen in the Table of Contents.

The arrangement into lessons by numbers in brackets — seventy-five for the first half year and seventy-five for the second half — is not to be considered as inflexible, but only as suggestive. It will be a good plan, however, to follow this rather closely, giving outside work when desirable.

Problems involving intricate and complex conditions have been avoided.

Optional Lessons are given for more advanced pupils or for review work. No new principles are presented in these; they are intended only for extra drill.

In "Original Work," the pupils should present papers showing neatness and care. These exercises will be useful as a test of the pupil's knowledge of the subject. By posting the best work, great emulation can be secured.

CONTENTS

FIRST HALF YEAR

SECOND HALF YEAR

GRAMMAR SCHOOL ALGEBRA

A FEW WORDS IN BEGINNING

ALGEBRA and arithmetic are two branches of mathematics that are very closely related, and the general principles of one apply to the other. The knowledge you have acquired in your work in *numbers* will be made the basis for your advancement in the new subject of algebra. In this way each step will seem easy, and you will be delighted with the facility with which algebra will help you over the difficult places in arithmetic.

In arithmetic you manipulate the numbers given, according to the conditions of the problem, to get your *answer*. In algebra, you start with a quantity representing that answer, and work backward until you reach the result required. It is as if you had the answer given and desired to prove it to be correct.

Algebra is arithmetic broadened and made general; in fact, it has been called "universal arithmetic." The one is no more difficult to understand than the other.

You will first learn how numbers are represented in algebra.

HOW QUANTITIES ARE REPRESENTED IN ALGEBRA

[1] Consider, for instance, the statement: "*Many* horses are fast." This does not state *how* many; that is, it is a general statement without regard to a particular number. In algebra, letters are used to represent a general number — any number. We can use a letter, as a or b or c, to represent any number.

Instead of saying "*Many* horses are fast,"

we can say "a horses are fast."

Likewise for "*Many* horses are large,"

we can say "b horses are large."

Again, for "*Several* cows are in a field,"

we can say "x cows are in a field."

Some of these, as a, are Jerseys, and some, as b, are Alderneys, and the rest, c, are Guernseys. Then we have a cows, and b cows, and c cows, or all together x cows in the field. The sum of a cows and b cows and c cows equals x cows; or, expressed algebraically,

$$a + b + c = x.$$

This statement is general and is true for any numbers.

1. 4 eggs, and 5 eggs, and 3 eggs are how many eggs?

   4 ones, and 5 ones, and 3 ones are how many ones?

4 a's, and 5 a's, and 3 a's are how many a's?
4 b's, and 5 b's, and 3 b's are how many b's?
4 x's, and 5 x's, and 3 x's are how many x's?

2. If an egg is equal in value to 3 ¢, how much are the eggs in Example 1 worth?

If $a = 4$ ¢, how much are the a's worth?
If $b = 6$ ¢, how much are the b's worth?
If $x = 10$ ¢, how much are the x's worth?

3. 7 eggs + 8 eggs + 3 eggs + 6 eggs are how many eggs?

7 ones + 8 ones + 3 ones + 6 ones are how many ones?
$7a + 8a + 3a + 6a$ are how many a's?
$7b + 8b + 3b + 6b$ are how many b's?
$7x + 8x + 3x + 6x$ are how many x's?

4. If the value of an egg = $2\frac{1}{2}$ ¢, how much are the eggs in Example 3 worth?

If $a = 4$, to how much are the a's equal?
If $b = 5$, to how much are the b's equal?
If $x = 2$, to how much are the x's equal?

5. 6 eggs + 7 eggs less 9 eggs less 2 eggs are how many eggs?

7 ones + 7 ones − 9 ones − 2 ones are how many ones?
$6a + 7a - 9a - 2a$ are how many a's?
$6x + 7x - 9x - 2x$ are how many x's?

6. If the value of an egg = 2 ¢, how much are the eggs that are left in Example 5 worth?

If $a = 5$, to how much are the a's that are left equal?
If $b = 6$, to how much are the b's that are left equal?
If $x = 1$, to how much are the x's that are left equal?

ORIGINAL WORK

Make up three examples like those on page 9, using other objects and other letters. Then assign values to the objects and letters and find the results.

You see that the same signs are used in algebra as in arithmetic.

[2] $$7 + 9 - \frac{8}{4} - 2 \times 3 = ?$$

You have probably learned in your arithmetic work that *operations of division and multiplication are to be performed on a quantity before the operations of addition and subtraction are performed.*

Thus, you first added 7 and 9; then you subtracted the quotient of $8 \div 4$. You did not subtract the dividend 8 from the sum of 7 and 9 and then divide by 4. This would have given you 2, instead of the right amount, 14. So in the case of $14 - 2 \times 3$, you must first multiply 2 by 3, then subtract the result, 6, from 14. This gives the correct amount, $14 - 6$, or 8.

What results do you get for the following?

1. $6 + 7 - 10 \div 2 + 4 \times 2 = ?$

2. $\frac{15}{5} + 4 \times 7 - 9 \times 3 + \frac{4 \times 6}{3} = ?$

3. $17 + \frac{25}{5} - \frac{8}{2} - 2 \times 4 = ?$

4. $7 + \frac{3 \times 8}{6} + 5 \times 4 = ?$

5. $4 \times 5 - \frac{7 \times 6}{3 \times 2} + 4 - 3 \times 5 + 4 \times 2 = ?$

6. $8 \times 3 + 4 \times \frac{6}{3} - 3 \times 4 \times 2 - \frac{8 \times 2}{2} = ?$

Would it make any difference in Example 6, if we should multiply 4 by 6 before dividing by 3? Try it. Why is the result the same?

7. $4 + \frac{6 \times 4 - 2 \times 3}{6} - \frac{5 \times 8 - 10}{5} = ?$

8. $\frac{3\frac{1}{2} \times 5\frac{1}{2}}{11} + \frac{3.5 \times 5.5}{5} - \frac{4 \times 2.7}{2} + .8 = ?$

[3] 1. Since there are 4 quarts in a gallon, how many quarts are there in 6 gallons? in 12 gallons?

2. Suppose we say there are x beans in a quart of beans. Then how many beans are there in 3 quarts? in 6 quarts?

3. If x represents a certain number, how many times that number are $3x$, $7x$, $12x$, ax, bx?

4. How much greater is $6b$ than $2b$?

5. How much greater is $12a$ than $8a$?

6. If x is a particular number, how should you represent $\frac{1}{2}$ of that number? $\frac{2}{3}$ of that number?

7. How should you express the difference between 10 and 6? 14 and 7?

8. How should you express the difference between $3a$ and a? $7b$ and $2b$? $3x$ and $2x$?

9. Indicate in full how you would express, without simplifying, the sum of 8 and 2 diminished by 3; the sum of 3 and 6 and 7 diminished by 5; the sum of 4 and 7 and 6 and a.

10. In the same way represent the sum of a and b; c and d; x and y; a and b and c; x and y and z; a and d diminished by y; a divided by b.

ORIGINAL WORK

[4] Make up three examples like the above.

PROBLEMS

1. If there are a apples in a barrel, how many apples are there in 7 barrels? in 12 barrels? in a barrels?

2. If a man earns b dollars in a day, how much will he earn in a week? in a month? in a year?

3. If a man earns m dollars a week, how much will he earn in b weeks?

4. How old were you 5 years ago? How old shall you be 10 years hence?

5. If a man is c years old, how old was he 3 years ago? 10 years ago? a years ago? How old will he be d years hence? e years hence?

6. If a man earns c dollars a day, how much will b men earn? x men?

7. A man gave a dollars for a suit of clothes, b dollars for a hat, and b dollars for a pair of shoes. How much did he spend?

8. If a huckster sold a bushels of apples at b cents a quart, how much did he get for all?

OPTIONAL WORK *

1. If a man bought a horse for a dollars and sold it for b dollars, did he gain or lose? Express the transaction. When would he gain? When would he lose?

2. A dealer bought d tubs of butter at a cents a pound. If each tub contains 50 pounds, how much did he pay?

3. A man has a dollars. If he owes b dollars to one man and c dollars to another, how much is he worth?

ORIGINAL WORK

[5] Make up five problems involving letters, and indicate the solution.

EQUATIONS

[6] 1. If $4x$ is equal to 20, how much does x equal?

2. If $3x = 9$, how much does x equal?

3. If $6y = 12$, how much does y represent?

4. If $7z = 28$, what is the value of z?

Find the value of x:

5. $2x = 6$.
6. $x + 4 = 7$.
7. $x = 9 - 3$.
8. $2x = 12 - 4$.
9. $3x + 7 = 22$.
10. $2x - 4 = 12$.

The above are called **equations**. An **equation** is an expression of equality between two quantities or sets of quantities.

ORIGINAL WORK

1. Make up an equation, using x, 9, and 3.

2. Make up three equations, using x, a, and b.

3. Make up three equations, using x and one or two figures, or other letters.

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* Optional work is intended for advanced pupils or for review work. (See Preface.)

PROBLEMS

[7] A farmer has 100 bushels of apples in two bins. In the larger one he has 3 times as many bushels as in the smaller one. How many bushels has he in each?

The solution by arithmetic would be about as follows:

Number of bushels in smaller bin + number in larger bin = total number bushels.

Number of bushels in larger bin = 3 times number in smaller bin.

Hence, number bushels in smaller bin + 3 times number bushels in smaller bin = total number of bushels.

Or, 4 times number in smaller bin = 100.
Number in smaller bin = 25.
Number in larger bin = 3 times 25, or 75.

Algebra enables us to express all such complex statements in simple form. Let us say:

Let x = number bushels in smaller bin.
Then $3x$ = number bushels in larger bin.
Or, $4x$ = number in both bins.

$$4x = 100.$$
$$x = 25, \text{ the number of bushels in the smaller bin.}$$
$$3x = 75, \text{ the number of bushels in the larger bin.}$$

Solve the following:

1. The sum of two numbers is 20, and 8 is one number. What is the other?

2. The sum of two numbers is 60, and the greater is 4 times the less. What are the numbers?

3. The sum of two numbers is 40, and the greater is 10 more than the less. What are the numbers?

4. The sum of two numbers is 30, and one number is twice as large as the other. What are the numbers?

5. Bryant had a roll of bills containing five-dollar bills and one-dollar bills. The value of the roll was $240. If there were as many one-dollar bills as five-dollar bills, how many of each kind were there?

6. At another time he had a roll of five-dollar bills, two-dollar bills, and one-dollar bills, there being the same number of each. If the value of the roll was $88, how many of each were there?

7. There are 600 persons in a crowd consisting of men and women, and there are twice as many men as women. How many of each are there?

8. If a vest costs $6 less than a coat, and if both cost $20, how much does each cost?

OPTIONAL WORK

1. A suit of clothes cost $21; the coat cost twice as much as the trousers, and the trousers twice as much as the vest. Find the cost of each.

2. Tom has 400 pens. He has in one box 100 more than twice the number in the other box. How many has he in each?

3. If a horse cost 4 times as much as the harness, and both cost $200, how much did each cost?

4. There are 150 cows on a dairy farm. There are 3 times as many in the middle field as in the small field, and twice as many in the large field as in the middle field. How many are there in each field?

ALGEBRAIC EXPRESSIONS

[8] The **terms** of an algebraic expression are the parts connected by the signs + or −. Thus,

In $a + b$ there are two terms.

In $a + b - c$ there are three terms.

In $a \times b \div c \times d - e \times f + x \div y$ there are three terms. Indicate them.

In the absence of a sign *before* a quantity in arithmetic or algebra, the + sign is understood; but the absence of a sign *between* quantities represented by letters has a different meaning in algebra from that in arithmetic.

Thus in the expression $a + x$, $7 + x$, quantity a is used additively, as is the 7.

But consider the expressions abc, 345. When signs are absent between letters, × is understood. Whereas 345 means $300 + 40 + 5$, abc means $a \times b \times c$.

What does xyz mean? 272?

THE PARENTHESIS

Is there any difference between $12+8\div 2$ and $8\div 2+12$? Between $6 + 3 \times 2$ and $3 \times 2 + 6$?

If we desire to add 12 and 8 in the first example before dividing by 2, we must have some way of showing it. The **parenthesis** ( ) is a device used for this purpose. It is used to inclose quantities that are to be treated as one quantity. Thus,

$(12+8)\div 2$ means $20\div 2$, and is different from $12+8\div 2$, which means $12 + 4$.

How should you write the expression $6 + 3 \times 2$, so that the result shall be 18 instead of 12?

Express the following by the proper signs:

1. Add 6 to 12, and divide the result by 3.
Add 6 to 12 divided by 3.
Is there any difference in the result? How much?

2. Subtract 10 from 24, and divide the result by 2.
Subtract 10 from 24 divided by 2.
Explain these expressions.

[9] 3. Add 4 to 8, and multiply the result by 3.
Add 4 to 8 multiplied by 3. Explain.

4. Add 6 and 4 and 5 and 3, and divide the result by 2.
What would be the result if we changed the expression to this: $6 \div 2 + 4 + 5 + 3$?

Or to this: $6 + 4 \div 2 + 5 + 3$?

Or to this: Add 6 and 4, and divide the result by 2; then add 5 and 3 to the result.

Find the value of each of the following expressions. In the four examples we have the same numbers, but the position of the parenthesis is different.

5. $(8 + 4 + 6 + 2) \div 2$.

6. $(8 + 4 + 6) + 2 \div 2$.

7. $(8 + 4) \div 2 + 6 + 2$.

8. $(8 + 4) \div 2 + (6 + 2)$.

Are the last two the same? Why?
What does the following expression mean?

$$(a + b + c + d) \div e.$$

How do the following expressions differ?

9. $3 \times 4 + 6$.
$3 \times (4 + 6)$.

10. $a \times b + c$.
$a \times (b + c)$.

COEFFICIENTS

[10] 3 apples and 4 apples and 2 apples are 9 apples. $3a$ and $4a$ and $2a$ are $9a$.

$3x$ and $4x$ and $2x$ are $9x$.

The 9 in each case shows how many times the quantity is taken additively. It is called the **coefficient.**

Point out the coefficients in $4x$, $5b$, $6d$, $14y$, $37m$.

In the expression ax, the a may be regarded as the coefficient of x, and means that x is to be added a times.

$$x + x + x + x + x \text{ and so on to } ax\text{'s} = ax.$$

In the expression $9ax$, 9 may be regarded as the coefficient of ax, or $9a$ as the coefficient of x.

Hence a **coefficient** is a figure or a letter placed before a quantity to show how many times it is taken additively.

When no coefficient is expressed, 1 is understood.

Write five quantities and name the coefficients.

EXPONENT

How can you express $3 \times 3 \times 3 \times 3 \times 3$ in one term? How many 3's are used? How are they used?

If we use 3 twice as a factor, we can express the result by 3^2, or by 9. 9 is called the second power of 3.

If we use 3 four times as a factor, we can express the result as 3^4. The product, 81, is the fourth power of 3.

What does 2^3 mean? a^3? 4^2? a^2? x^2?

The **power** of a quantity is expressed by a small figure or by a letter, called an **exponent,** written a little above the quantity at the right.

When no exponent is written, the exponent 1 is understood.

a^2 is read: "a square."

a^3 is read: "a cube."

a^4 is read: "a fourth power."

a^m is read: "a exponent m."

$a^{\frac{1}{2}}$ is read: "a exponent $\frac{1}{2}$."

a^{-2} is read: "a exponent -2."

Read the following and state what each means:

$$4^3,\ a^4,\ b^8,\ x^2,\ y^2,\ a^3,\ b^2,\ e^2,\ 3\,a^2,\ 6\,b^2;$$
$$a^2+b^2,\ 2\,a^3-b^3,\ a^2+7\,b^2-c^2+d^2;$$
$$ax^2+bcy-abcd.$$

ORIGINAL WORK

Write ten quantities and name the coefficients and the exponents.

POSITIVE AND NEGATIVE QUANTITIES

[11] Positive and negative terms are those which are *opposite* in character.

Thus we regard a man's property, or his assets, as positive because they increase his wealth, and his debts as negative, because they decrease his wealth.

If the direction north is called positive, the opposite direction, south, is called negative.

Up is generally regarded as a positive direction, and down as a negative direction. Thus on a thermometer the degrees above zero are marked + and below zero, −.

Positive quantities are represented by the + sign.

Negative quantities are represented by the − sign.

Place the proper signs before the quantities in the following examples to indicate their true relation:

1. A man had property worth $10,000, and owed $2000. How much was he worth?

2. A boy ran 100 yards east, and then 50 yards west.

3. A man had $50, and spent $20.

4. A man had $10, earned $15, and spent $5.

5. A man had 50 sheep. He bought 20, and sold 30. He then bought 60, and sold 70. How many had he?

ORIGINAL WORK

Make up three problems illustrating positive and negative quantities.

[12] The value of an expression is the same, no matter in what order the terms are written.

Thus, $8 + 7 - 3 + 4 - 6$

is the same as $8 - 3 + 4 - 6 + 7.$

And $a + b + c - d - e$

is the same as $-d + a - e + b + c.$

Rewrite these in another way, remembering that the sign before a term belongs to the term.

Write three expressions and then change the position of the terms.

If no sign is expressed, + is always understood. Thus in the expression a the sign is +, the coefficient is 1, and the exponent is 1.

Similar or **like terms** are terms that have the same letters with the same exponents. All others are **unlike** or **dissimilar.**

Thus: $7\,a^2,\ -3\,a^2,\ +6\,a^2$ are similar terms.

$3\,a^2b,\ 2\,a^2b^2,\ 7\,a^2c^2$ are dissimilar or unlike terms.

ORIGINAL WORK

Write five similar terms; five dissimilar terms.

[13] An algebraic expression of one term is called a **monomial.**

An algebraic expression of two terms is called a **binomial.**

An algebraic expression of three terms is called a **trinomial.**

An algebraic expression of two or more terms is called a **polynomial.**

Are binomials and trinomials polynomials? Why?

Classify the following expressions as above:

1. $3\,a^2$.
2. $2\,a+4\,b+c$.
3. $2\,x+3\,y$.
4. $ax+bx+cx$.
5. $2\,z+4\,ab+6\,c-8\,dx$.
6. $2\,a\div c+7\,xy-3\,x\div y$.

ORIGINAL WORK

1. Write two similar terms.

2. Write a binomial having two similar terms.

3. Write a trinomial with one positive and two negative terms.

4. Write a polynomial of five terms, four being similar, and three positive and two negative.

5. Write a polynomial of six terms, all positive and similar. Can you express it as a monomial?

SUBSTITUTION

[14] If $a=1$, $b=2$, $c=3$, $d=4$, $e=5$, $f=6$, $m=0$, $n=\frac{1}{2}$, $x=8$, $y=10$, $z=100$, find the value of each of the following:

1. $4d$.
2. $3b^3$.
3. $4m$.
4. $7y$.
5. $6e^2$.
6. x^2.
7. $2c$.
8. $6n$.
9. $14y$.
10. $2y^2$.
11. $2d^2$.
12. $7a^2$.
13. $2d^3$.
14. $6f^2$.
15. $3c^2$.

[15]

16. $3x$.
17. $2z$.
18. $7d$.
19. $9c^2$.
20. $4a$.
21. $2e^4$.
22. $7y^3$.
23. n^2.
24. $8n^3$.
25. $\frac{1}{8}c^3$.
26. $a+b+c+d$.
27. $bx+cd$.
28. $3abc$.
29. $2an$.
30. $\frac{y^2}{10}$.
31. $abc+dx$.
32. c^2-bd.
33. $2b^2+3c^2$.
34. $5c^2+b^2$.
35. $\frac{y^2}{b}$.

[16]

36. $4a+3d+c^2+x^2+f^2-z$.
37. $4n+c^2-2e$.
38. b^3+nd^2.
39. $4a^4+7x+2y$.
40. $adn+em-\frac{y}{2e}+\frac{x}{ad}+f$.
41. $ax+ny-\frac{nz}{5e}-c^2$.
42. $m(a^2+b^2+c^2+e^2+x^2)$.
43. $(a+e)-(f-d)$.
44. $nx-\frac{y}{e}$.
45. $b(a+d)$.
46. $y(z-12x)$.
47. $z+(y^2+3z+10y)$.

[17] If $a=2$, $b=4$, $c=6$, $d=8$, $x=1$, $y=3$, $z=5$, $n=\frac{1}{2}$, find the value of:

48. $3\,ax$.

49. $\frac{d}{a}$.

50. $\frac{c^2}{a^2}$.

51. $\frac{3\,y}{3\,cn}$.

52. $\frac{d^2}{b^2}$.

53. $\frac{4\,y}{b}$.

54. $\frac{2\,b^2}{ad}$.

55. $\frac{d+x}{y}$.

56. $\frac{a+c}{c-a}$.

57. $\frac{a(d+b)}{cn}$.

58. $\frac{z+y}{bn}$.

59. $\frac{2\,a+4\,x}{dn}$.

ORIGINAL WORK

Make up three examples using the values given above.

$? = 25$. $? = 12$. $? = 10$.

OPTIONAL WORK

Using the values given above, find the values of:

1. $a^2b^2c^2n^2 - x^2y^2z^2$.

2. $z^2 + c^2 - 2\,cn$.

3. $\frac{a}{d} + \frac{x}{b} + \frac{n}{a} + \frac{y}{bcn}$.

4. $\frac{2\,an}{4\,bn^2} + \frac{5\,b}{dz}$.

5. $\frac{ac - bn}{adn - bny}$.

6. $\frac{y(a^2 + c^2)}{2(b^2 - 3\,a^2)}$.

7. $\frac{xyz + abcn}{cy - x(d - a - x)}$.

8. $\frac{x^2 + abn^2}{a^2n^2 + dn^2}$.

If $x=2$, $y=5$, $z=4$, find the values of:

9. $\frac{x^2(y^2 - z^2) - z(x+z)}{y - x + z^2 - 3}$.

10. $\frac{x^2 + 2\,xy + y^2}{x^2 - 2\,xy + y^2}$.

ADDITION

[18] **1.** How many nuts are 6 nuts and 4 nuts?

How many a's are $7\,a$ and $6\,a$ and $4\,a$ and $3\,a$?

How many xy's are $2\,xy$, xy, $3\,xy$, and $4\,xy$?

Add $4\,b$, $7\,b$, $8\,b$, and $2\,b$.

What did you learn on page 19 about positive and negative quantities and their signs?

2. A man had \$20. He earned \$13 one week, \$12 another, \$15 the third, and \$14 the fourth week. He spent \$10 for rent, \$20 for food, \$4 for coal, and \$12 for clothes. How can we represent his financial standing? What he had and what he earned evidently increase his wealth; hence are positive. The sums he spent decrease his wealth, and are negative. Therefore he has,

$$\$20+\$13+\$12+\$15+\$14-(\$10+\$20+\$4+\$12),$$

or

$$\$74-\$46=\$28.$$

3. Again, a man earned during 4 weeks \$8, \$10, \$12, and \$12 respectively. He bought as follows: groceries \$16, coal \$10, clothes \$14, and paid \$8 for rent. If what he earns is positive and what he owes is negative, we have

$$\$8+\$10+\$12+\$12-\$16-\$10-\$14-\$8,$$

or

$$\$42-\$48=-\$6.$$

What does this signify? How shall we represent it?

How many are:

4. 3 apples and 4 apples?

5. $2\,a$ and $5\,a$?

6. 3 apples and 4 pears?

7. $3\,a$ and $4\,b$?

Which of these examples can be expressed in a single term? What kind of terms have they? (See page 20.)

Addition is the process of combining two or more quantities into the simplest expression, called their sum.

Only similar terms can be united into one term by addition, but with dissimilar terms the addition may be expressed. Thus, x and y cannot be united into one term by addition, but their sum may be expressed as $x + y$.

[19] I. Similar terms having the same sign.

Add:

1.	2.	3.	4.
$2a$	$4y$	$4xy$	$2ax$
$3a$	$3y$	$3xy$	$3ax$
$4a$	$6y$	$4xy$	$9ax$
$5a$	$2y$	$2xy$	$2ax$

5.	6.	7.	8.
$7y^2$	$3a^2$	$6y^2z^2$	$9abc$
$2y^2$	$4a^2$	$4y^2z^2$	$6abc$
$3y^2$	$5a^2$	$2y^2z^2$	$4abc$
$8y^2$	$9a^2$	$8y^2z^2$	$6abc$

9.	10.	11.	12.
$-3a$	$3(a+b)$	$-9xy$	$-6(x-y)$
$-4a$	$4(a+b)$	$-4xy$	$-8(x-y)$
$-5a$	$(a+b)$	$-6xy$	$-3(x-y)$
$-6a$	$6(a+b)$	$-7xy$	$-2(x-y)$

What kind of terms are in each of the above examples?
What are the figures before the letters called?
What can you say of the signs?
What is the sign in the sum?
Make a rule for adding similar terms with the same sign.

ORIGINAL WORK

Bring in five examples in addition of similar terms with like signs.

[20] II. Similar terms with different signs.

Add:

1.	2.	3.	4.	5.
14	-12	16 dollars	9 ¢	$8c$
6	-6	7 dollars	7 ¢	$5c$
-7	18	-3 dollars	-6 ¢	$-3c$
-3	-5	-6 dollars	-12 ¢	$-7c$

6.	7.	8.	9.	10.
$3ab$	$6xy$	$6a^2b^2$	$-4xy^2$	$2(a+b)$
$-2ab$	$-4xy$	$-7a^2b^2$	$-3xy^2$	$3(a+b)$
$-4ab$	$-xy$	$2a^2b^2$	xy^2	$7(a+b)$
$7ab$	$-3xy$	a^2b^2	xy^2	$-9(a+b)$

Write as above and find sum:

11. $6a+4a-7a-2a$.

12. $2c+9c-3c-c-c+c$.

13. $7xy-7xy+3xy-xy$.

14. $4a^2c^2-a^2c^2+7a^2c^2+3a^2c^2$.

15. $9ab^2+7ab^2+16ab^2-10ab^2-4ab^2-ab^2$.

What kind of terms are in each of the above examples from **1** to **15**?

What can you say of the signs?

How did you find the sum?

ORIGINAL WORK

[21] Make up three examples like those on page 26, and find their sums.

III. Polynomials.

Add:

1.
$$\begin{array}{r} a+\ \ b+\ \ c \\ 3a+2b-2c \\ 3a+4b-\ \ c \\ \underline{2a-6b+3c} \end{array}$$

2.
$$\begin{array}{l} 6b-4c \\ 3b+4c-\ \ d \\ 7b-6c-4d \\ \underline{2b-6c-\ \ d} \end{array}$$

3.
$$\begin{array}{c} a \\ \underline{b} \end{array}$$

4.
$$\begin{array}{c} a \\ b \\ \underline{c} \end{array}$$

5.
$$\begin{array}{r} a \\ -b \\ -c \\ \underline{-d} \end{array}$$

6.
$$\begin{array}{r} ax+\ \ ay\qquad\ \ \\ 3ax-3ay-3az \\ 2ax-2ay-2az \\ \underline{-4ax+4ay+4az} \end{array}$$

7.
$$\begin{array}{r} 2a+3(b+c) \\ 4a-4(b+c) \\ -6a+2(b+c) \\ \underline{-2a-5(b+c)} \end{array}$$

[22] Write as above and add. Be careful to write similar terms in the same column.

8. $a+3b-2c+2a+2b+c+4a-2b+2c.$

9. $ax-by-2d+2ax-3by+4d+4ax+2by-3d.$

10. $3a^2-4b^2-2c^2-4a^2+2b^2-6c^2+4a^2+2b^2+2c^2.$

11. $2x^3+2y^3-3z^3-\frac{1}{2}x^3-\frac{1}{2}y^3-\frac{1}{2}z^3+2\frac{1}{2}x^3-1\frac{1}{2}y^3+\frac{1}{2}z^3.$

12. $3ab+7ab^2+4a^2b-2ab-6ab^2-2a^2b+5ab+3a^2b-7ab^2.$

What kind of algebraic expressions are the above?
What must be done before you can add them?
How do you add polynomials?

ORIGINAL WORK

Make up two examples like the above and find their sums.

[23] **IV. Dissimilar terms with a common part.**

Add $7a$ and $6a$.

The sum is $13a$ or $(7+6)a$.

The latter expression is not in its simplest form.

Express in the same way:

1. $6c+4c$.
2. $7d+2d$.
3. $3xy+2xy$.
4. $4cd+7cd$.

Add ax and bx.

In this case a and b are to be regarded as the coefficients of x.

We have a x's and b x's, which, as above, are $(a+b)x$.

Add in the same way:

1.	2.	3.	4.
ac	ax	ay^2	ax
bc	bx	$-by^2$	x
	cx	cy^2	

5.	6.	7.	8.
$a(x+y)$	$m(a+b)$	$4(x-y)$	a^2m
$b(x+y)$	$-4(a+b)$	$a(x-y)$	b^2m
		$b(x-y)$	c^2m

PROBLEMS

[24] 1. A boy had a marbles. He won b marbles from George, and c marbles from Fred, and d marbles from Bert. Express how many he then had.

2. A man had a dollars and earned b dollars. He found c dollars, and had b dollars given to him. How many dollars had he then?

3. A man had a horses. He bought b horses of one man, and a horses of another. How many had he then?

4. Frank is d years old. Fred is twice as old as Frank; and their father is twice as old as both together. How old is their father?

5. A lady spent c dollars for a dress and b dollars for a hat, c dollars for shoes, and as much for a parlor suit as she spent for all of these. How much did she spend for a parlor suit, and how much for all?

6. A man bought a carriage for a dollars. He gave 3 times as much for a horse, and $\frac{1}{2}$ as much for a harness. How much did they all cost?

7. A wholesale merchant had a carload of sugar. He sold a barrels to one man, 9 barrels to another, 6 barrels to another, and d barrels to a fourth. How many barrels did he sell?

8. A man began business with $2a$ dollars. The first year he gained b dollars, and the second year he lost a dollars. How much did he then have?

9. Frank rode a miles on the cars. He then rode b miles on his bicycle and walked c miles. How far did he travel?

10. I paid m dollars for my fare to Chicago, n dollars for a sleeper, and p dollars for meals. My other expenses were r dollars. How much did I spend?

11. A man bought a wagon for m dollars, a harness for n dollars, and a horse for the price he paid for both. He sold the whole outfit at a gain of s dollars. How much did he get for it?

OPTIONAL WORK

Add:

1. $3\,ax + 7\,ay + 4\,az$, $3\,ax - 4\,ay - 3\,az$, $9\,ax + 2\,ay + 4\,az$, $2\,az - 12\,ax - 6\,ay$.

2. $3\,a + 7\,b - 2\,c$, $4\,a - 9\,b + 3\,d$, $6\,c - 4\,a + 4\,c - 2\,d$, $3\,b + 4\,c - d$, $7\,a + 6\,b - 3\,c$.

3. $3\,x + 4\,z + 8\,y$, $2\frac{1}{2}\,x - 1\frac{1}{2}\,z - 4\frac{1}{2}y$, $3\,x - 6\,z - \frac{1}{2}\,y$, $3\frac{1}{2}\,z - 5\frac{1}{2}\,x - 3\,y$.

4. $4\,a^2x + 6\,by^2 - 2\,x^2y^2$, $2\,a^2x - 4\,x^2y^2 - 5\,by^2$, $2\,x^2y^2 + 4\,a^2x - 3\,by^2$.

5. $6\,c^2 - 4\,c^3 - 3\,c + 4$, $9\,c^3 + 4\,c - 8$, $7\,c^2 - 6\,c + 12$.

6. $ax + dy + xy$, $bx + cy + 3\,xy$.

7. $2(a + b) + 3(b + c)$, $4(a + b) + 5(b + c)$, $2(a + b) - 9(b + c)$.

8. $4(x - y) - 3(x + y) - b$, $b - 3(x - y) + 4(x + y)$, $3\,b + 3(x - y) + 2(x + y)$.

9. $\frac{1}{2}\,ax + \frac{1}{3}\,xy + \frac{1}{2}\,z$, $\frac{1}{3}\,ax - \frac{1}{2}\,xy - \frac{1}{3}\,z$, $\frac{1}{2}\,xy - \frac{1}{4}\,ax + \frac{1}{2}\,z$, $\frac{1}{4}\,ax + \frac{1}{4}\,z - \frac{1}{3}\,xy$.

10. $\frac{1}{3}\,a^2 - \frac{3}{8}\,b^2 + \frac{1}{4}\,c^2$, $\frac{1}{2}\,a^2 + \frac{1}{4}\,b^2 - \frac{1}{2}\,d$, $\frac{3}{8}\,a^2 - \frac{1}{3}\,c^2 + \frac{1}{4}\,d$, $\frac{3}{8}\,b^2 - \frac{5}{12}\,a^2 - \frac{1}{2}\,c^2$, $\frac{1}{3}\,d - \frac{1}{4}\,a^2 + \frac{1}{8}\,b^2$.

ORIGINAL WORK

Make up two problems in addition like the above and solve them.

SUBTRACTION

[25] On page 19 you learned something about positive and negative quantities. You learned that these quantities are represented by the signs + and −.

You have also dealt with these in addition.

In arithmetic you dealt only with positive quantities; that is, quantities greater than zero.

In algebra you will deal with quantities *greater* than zero and quantities *less* than zero.

Subtraction is the process of finding the difference of two quantities.

The **minuend** is the quantity from which the other is subtracted.

The **subtrahend** is the quantity to be subtracted.

The **difference** or **remainder** is the quantity left after subtracting.

Take the series of numbers above zero, as follows:

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, etc. These are all positive quantities.

Now, if we extend the series below zero, it must represent the quantities "less than zero," or negative, or −. We shall have a series from −10 up through 0 to +10, as

−10, −9, −8, −7, −6, −5, −4, −3, −2, −1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10.

If we go toward the *right*, the values increase, that is, we *add;* if toward the *left*, they decrease, that is, we *subtract*.

Let us apply this idea of positive and negative direction to the solution of a few examples.

$-10, -9, -8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10.$

I. **Subtract 2 from 8.**

As no sign is expressed, + is, of course, understood before the 2 and the 8.

In the above series, if we subtract 2 from 8, we must move in the negative direction, to the left, 2 places from 8, and we reach + 6. Hence 2 from 8 equals 6.

II. **Subtract 8 from 2.**

To subtract 8 from 2, we move 8 places to the left from 2 and reach − 6. Hence 8 from 2 equals − 6.

III. **Subtract 2 from − 8.**

To subtract 2 from − 8, we move two places to the left of − 8 and reach − 10. Hence 2 from − 8 equals − 10.

IV. **Subtract 8 from − 2.**

To subtract 8 from − 2, we move eight places to the left of − 2 and reach − 10. Hence 8 from − 2 = − 10.

Can you explain why III and IV give the same results?

V. **Subtract − 2 from 8.**

If we subtract 4 from 8, the result is 4; if we subtract 3 from 8, the result is 5; 2 from 8, the result is 6; 1 from 8, the result is 7; 0 from 8, the result is 8; − 1 from 8, the result must be 1 more than the result when 0 is subtracted from 8, or 9, for − 1 is 1 less than 0; − 2 from 8, the result is 10. That is, when we *subtract* a negative quantity (a quantity less than zero), the result is the same as *adding* a positive quantity.

Hence, to subtract − 2 from 8, we must go the *right* in a positive direction two places from 8, and we reach 10. Hence − 2 from 8 equals 10.

VI. **Subtract − 8 from 2.**

To subtract − 8 from 2, we move eight places to the right from 2 (as explained in V) and reach 10. Hence − 8 from 2 equals 10.

VII. **Subtract -2 from -8.**

To subtract -2 from -8, we move two places to the right from -8 and reach -6. Hence -2 from -8 equals -6.

VIII. **Subtract -8 from -2.**

To subtract -8 from -2, we move eight places to the right from -2 and reach $+6$. Hence -8 from -2 equals $+6$.

Let us arrange these eight examples in the form of subtraction.

Subtract:

I.	II.	III.	IV.	V.	VI.	VII.	VIII.
8	2	-8	-2	8	2	-8	-2
2	8	2	8	-2	-8	-2	-8
6	-6	-10	-10	10	10	-6	6

In algebra, as in arithmetic, the difference is that quantity which when added to the subtrahend will equal the minuend.

It will readily be seen in each of the above eight examples that the difference could have been found by *changing the sign of the subtrahend and adding it to the minuend.*

[26] Write a series of integers from -20 to $+20$, and solve the following eight examples.

Subtract:

1.	**2.**	**3.**	**4.**	**5.**	**6.**	**7.**	**8.**
16	3	-12	-6	13	5	-19	-2
4	11	7	8	-6	-14	-4	-10

Now rewrite Examples 1 to 8 on page 33, and change the signs of each subtrahend and add it to its proper minuend. Are the results the same?

[27] Inspect the following eight examples, and see whether the difference can be found by changing the sign of the subtrahend and adding it to the minuend.

1.	2.	3.	4.	5.	6.	7.	8.
$11x$	$4x$	$11x$	$-4x$	$4x$	$-11x$	$-4x$	$-11x$
$4x$	$11x$	$-4x$	$11x$	$-11x$	$4x$	$-11x$	$-4x$
$7x$	$-7x$	$15x$	$-15x$	$15x$	$-15x$	$7x$	$-7x$

RULE. *Write the subtrahend under the minuend with similar terms in the same column. Consider the sign of each term of the subtrahend to be changed from + to − and − to +, and add to the minuend.*

Copy the following examples, and find the difference by applying the above rule.

Test your work by adding the difference to the subtrahend and seeing whether the result equals the minuend.

Subtract:

	1.	2.	3.	4.	5.
I.	$6a$	$8b$	$14x$	$24y$	$14m$
	$2a$	$3b$	$3x$	$6y$	$3m$

	1.	2.	3.	4.	5.
II.	$3x$	$2a$	$3b$	$4c$	d
	$7x$	$9a$	$16b$	$10c$	$9d$

	1.	2.	3.	4.	5.
III.	$-4a$	$-17a^2$	$-13a^3$	$-27a^4$	$-16a^5$
	$2a$	$7a^2$	$9a^3$	$13a^4$	$3a^5$

	1.	2.	3.	4.	5.
IV.	$-3c$	$-9ab$	$-6m^2$	$-2xy$	$-x^2y^2$
	$8c$	$14ab$	$7m^2$	$5xy$	$8x^2y^2$

[28]	1.	2.	3.	4.	5.
V.	$7m$	$12c$	$4ab$	$9a^2$	$4d$
	$-3m$	$-7c$	$-ab$	$-2a^2$	$-d$

	1.	2.	3.	4.	5.
VI.	$12a^2$	$2ac$	$4xy$	z	$19z^2$
	$-17a^2$	$-6ac$	$-9xy$	$-4z$	$-25z^2$

	1.	2.	3.	4.	5.
VII.	$-8m^5$	$-12m^4$	$-7m^3$	$-3m^2$	$-9m$
	$-m^5$	$-2m^4$	$-m^3$	$-m^2$	$-8m$

	1.	2.	3.	4.	5.
VIII.	$-7bx$	$-9b^2x^2$	$-6b^3x^3$	$-12b^4x^4$	$-b^5x^5$
	$-13bx$	$-17b^2x^2$	$-9b^3x^3$	$-14b^4x^4$	$-5b^5x^5$

MISCELLANEOUS EXAMPLES IN SUBTRACTION

[29] (Be careful to write only similar terms in the same column.)

Subtract: 1. $9ax$ from $17ax$.

2. $3ay + 4b$ from $6ay - 4b$.

3. $3a + 4b - c$ from $9a - 7b + c$.

4. $6x + 5$ from $9x - 17$.

5. $2(x + y)$ from $6(x + y)$.

6. $6a^2b^2 + 4ab + b^2$ from $6a^2b^2 - 4ab - b^2$.

7. $a^2 - 2ab + b^2$ from $a^2 + 2ab + b^2$.

8. $12a^2 - 17a^3x^3 + 4ax$ from $10a^2 + 6ax - 12a^3x^3$.

9. $a^3 + 3a^2b + 3ab^2 + b^3$ from $b^3 - 3ab^2 + 3a^2b - a^3$.

10. $4m^3 + 6m^2n - 9n^2$ from $4m^3 + 6m^2n - 9n^2 + 7$.

11. From $8x^3 + 9xy + 7x^2 - 3y^2$ take $3x^2 + 7y^2 + 6x^3 - 9xy + 10$.

12. From $6x + 4$ take $7 - 6x + 3y$.

[30] 13. From $a^3 + b^3$ take $a^3 + 3a^2b + 3ab^2 + b^3$.

14. From $d^3 + 320a - 140m + 16n - 100a^2$ take $160a - 40m + 3d^3 + 7n + 100a^2$.

15. From $\frac{1}{2}m^2 - \frac{1}{4}n^2 + \frac{1}{2}$ take $\frac{3}{4}n^2 - \frac{1}{2}m^2 + \frac{1}{2}$.

16. From the sum of $a^2 + 2ab + b^2$ and $a^2 - 2ab + b^2$ take $a^2 + b^2$.

17. From the sum of $6a^2 - 4ab + 7m + 9$ and $7 + 6ab - 7m - 3a^2$ take $2a^2 - 6ab - 6 + 9m$.

18. From the sum of $3 + 9a + 7b + 3n + 6d$ and $4d + 3n + 9b + 7 + 4c$ take $6a + 4 + 6c - b + 6n$.

19. From the sum of $a^3 + 3ab^2 + b^3$, $a^3 - 3a^2b + 3ab^2 - b^3$, and $2a^3 + 6ab^2 + 6b^3$ take $a^3 + 3ab^2 + 2b^3$.

20. From the sum of $2m^2 + 4mn + 2n^2$, $n^2 - 4mn - m^2$, $3m^2 - 3n^2$, and $m^2 + 3n^2$ take the sum of $4mn + 6n^2 - 2m^2$ and $2m^2 - 4mn - 3n^2$.

21. Take the sum of $7a + 9b + 16c - 17$ and $7 - 8c + b - 8a$ from $6c - 10 + 4b + a$.

PROBLEMS

[31] 1. Fred had a marbles. He won b marbles from James, d marbles from Will, and $3a$ marbles from Charles. He lost c marbles to Alfred, $2b$ marbles to Frank, and $4d$ marbles to Edward. How many had he then?

2. A farmer had a sheep in one field, b in another, and d in a third. He sold c sheep to one man, e sheep to another, and 65 to a third. After that, if 5 died, how many had he left?

3. If the sum of two numbers is 10 and one number is a, what is the other?

4. If the sum of two numbers is 5, how may the numbers be represented?

5. If the difference of two numbers is a, how may the numbers be represented?

6. A man earns a dollars a day and spends b dollars. How much has he at the end of the week?

7. The janitor of a building earns c dollars a month and spends d dollars. How much has he at the end of a year?

8. A man earns m dollars a month and receives n dollars a month interest on an investment. If he spends a dollars for rent and d dollars for household expenses, how much has he at the end of a year? at the end of 4 years?

OPTIONAL WORK

1. Take ax from bx.
2. Take cy from dy.
3. From ay take dy.
4. From $ay + by$ take cy.
5. From $2x + 4x$ take ay.

THE PARENTHESIS

[32] On page 16 you learned that the parenthesis ( ) is used to inclose quantities that are to be treated as one quantity. Hence a quantity in a ( ) with a + sign before it is to be taken additively; but if it has a − sign before it, it is to be taken subtractively.

Thus $(6a+4b-c)-(4a+4b-3c)$ is merely an indication that the quantity in the second parenthesis is to be subtracted from the quantity in the first.

Since in subtraction we change the signs of the subtrahend and add, the above example becomes:

$$6a+4b-c-4a-4b+3c \text{ or } 2a+2c.$$

Hence the following principle:

PRINCIPLE. *A parenthesis preceded by the − sign may be removed from an expression if the signs of all the terms in the parenthesis are changed.*

Remove the parenthesis:

1. $16a^2-(4a^2+6a^2-9a^2)$.
2. $7ab-(3ab+4b^2)$.
3. $1-(3a^2+2ab+b^2-6)$.
4. $a-b-(3a-b)$.
5. $(3a+4b)-(4a-6b)$.
6. $3a+4b-(4a-6b)$.
7. $-(3a+4b)-(4a-6b)$.
8. $(3a+4b)+(4a-6b)$.
9. $3m+4n-(2m-4n+6m^2)$.
10. $7ac-(3bd-6ac-4a^2)$.

ORIGINAL WORK

Make up two examples like the above and remove the parenthesis.

MULTIPLICATION

[**33**] As you have learned in arithmetic, multiplication is a short method of addition when the quantities to be added are the same.

How many are:

$$3+3+3+3+3? \qquad a+a+a+a+a?$$

How many are $3a+3a+3a+3a+3a$, or how many are 5 times $3a$?

How many are 10 times $3a$? 3 times $2b$?

How many are $-2a-2a-2a-2a$, or 4 times $-2a$?

How many are 6 times $-3b$? 4 times $-4b$?

The multiplier and the multiplicand may be positive or negative.

Is there any difference between 3×5 and 5×3? 7×9 and 9×7? $3\times 4\times 6$ and $4\times 6\times 3$, or $6\times 3\times 4$?

In the same way, no differences can arise in the product of quantities in algebra by changing the *order* of the factors. The product resulting by taking a b times must be the same as the product resulting by taking b a times, or a times b is the same as b times a.

The product of factors is the same in whatever order they are taken.

Multiplication is the process of taking one quantity as many times as there are units in another.

The **multiplicand** is the quantity to be taken.

The **multiplier** is the quantity by which we multiply.

The **product** is the result of the multiplication.

I. Both factors may be positive; as, $+a \times +b$.

II. One may be negative and the other positive; as, $-a \times +b$, or $+a \times -b$.

III. Both may be negative; as, $-a \times -b$.

I. When both factors are $+$.

Multiply 4 by 5.

This means that 4 is to be added five times. The result is $+20$.

Multiply b by a.

This means that b is to be added a times.

Since, if b is added three, four, five, six, or any number of times, the results will be $3b$, $4b$, $5b$, $6b$, or any number of b's; if $+b$ is added a times, the result will be $+ab$, a *positive result.*

II. When one factor is $+$ and the other $-$.

Multiply -4 by $+5$.

This means that -4 is to be added five times. The result is -20.

Multiply $-b$ by a.

This means that $-b$ is to be added a times.

2, 3, 4, 5, 6, or any number of $-b$'s will give $-2b$, $-3b$, $-4b$, $-5b$, etc., and a number of $-b$'s will give $-ab$.

Multiply 4 by -5.

4 multiplied by 5 indicates that 4 is to be *added* five times, and 4 multiplied by -5 indicates that 4 is to be *subtracted* five times.

If we subtract 4 once, we shall have -4. If we subtract 4 five times, we shall have -20. This is the same as taking 4 five times and subtracting the product.

Multiply b by $-a$.

$b \times a = ab$, but $b \times -a$ signifies that the product ab is to be subtracted. Hence, $b \times -a = -ab$.

III. When both factors are $-$.

Multiply -4 by -5.

In Case II we learned that $-4 \times 5 = -20$, and that the $-$ sign before the multiplier signifies that the multiplicand is to be taken five times subtractively. If we subtract -4 once, we shall have $+4$. If we subtract -4 five times, we shall have $+20$. Hence,

$$-4 \times -5 = -(-20) = +20.$$

Multiply $-b$ by $-a$.

In Case II we learned that $-b \times a = -ab$.

In $-b \times -a$, the sign before the multiplier signifies that the product $-ab$ is to be subtracted. If we subtract the $-ab$, or $-(-ab)$, we have $+ab$. Hence, $-b \times -a = +ab$.

We see from the above that when the signs of multiplicand and multiplier are alike, the product is $+$; when they are unlike, the product is $-$; or, we may remember this:

Like signs produce $+$ and unlike signs $-$.

Write the products of:

1.	**2.**	**3.**	**4.**	**5.**	**6.**
7	7	-7	-7	8	-8
4	-4	4	-4	-6	-8

7.	**8.**	**9.**	**10.**	**11.**	**12.**
-6	$-a$	$-m$	$-x$	y	$-y$
-8	$-b$	$-n$	y	$-x$	$-d$

[34] **13.** Multiply $3a$ by $2b$.

As the product of several factors is the same in whatever order they occur,

$$3a \times 2b = 3 \times 2 \times a \times b, \text{ or } 6ab.$$

We have learned that an exponent is used to indicate how many times a letter is to be taken as a factor. Hence,

$$a \times a = a^2.$$
$$a^2 \times a = a \times a \times a = a^3.$$
$$b^3 \times b^2 = b \times b \times b \times b \times b = b^5.$$
$$x^3 \times x^4 = x \times x \times x \times x \times x \times x \times x = x^7.$$

From the examples above we see that the exponent of a quantity in a product is equal to the **sum** of its exponents in the factors.

RULE. *To multiply two monomials, multiply the coefficients together for the coefficient of the product. To this coefficient annex the different letters, giving each letter an exponent equal to the sum of the exponents of that letter in the factors.*

If the signs are alike, the product is $+$; *if unlike,* $-$.

Multiply:

14.	**15.**	**16.**	**17.**
$3a$	$9b$	$-2x$	$2y$
$7a$	$-2b$	$-3x$	$-6y$
18.	**19.**	**20.**	**21.**
$3y$	$4x^2$	$7ay$	$-2m^2$
$-4y$	$-7x^2$	$3ax$	$-3n^2$
22.	**23.**	**24.**	**25.**
$-9xy^2$	$12abc$	$16a^2xy$	$75c^2d^5$
$6ay^3$	$-3a^2b^2d^2$	bc	$-2a^2c^2d^3$
26.	**27.**	**28.**	**29.**
$7b^2c^2$	$2ax$	$-a^2b^3c^4d^5$	ayz^3
$-2a^2c^3$	$-2by$	$-abcd$	$-a$

[35]

30.	31.	32.	33.
$-14\,p^2a^3z^{10}$	$-3\,xy^2$	$-8\,x^3y^3z^3$	$4\,c^2d^4$
$-2\,p^8x^7z$	$-y^3x$	$-2\,ab^2c^2$	$-4\,c^5d^3$
34.	**35.**	**36.**	**37.**
$-5\,xy$	$2\,axy^5$	$2(a+b)^2$	$4(a-c)^3$
$-5\,x^9yz$	$-6\,ay^6z$	$3(a+b)^3$	$-6(a-c)$
38.	**39.**	**40.**	**41.**
$-5(x+y)^2$	$3(x-y)$	$7(a-b)$	$4(x+y)$
$5(x+y)^4$	$4(x-y)$	$-4(a-b)$	$5(x+y)^2$

In these examples the quantity within the parenthesis is to be treated as one quantity.

ORIGINAL WORK

Make up five examples under this case and find the product.

[36] When the multiplicand is a polynomial.

1. Multiply 236 by 4.

$$236 = 200 + 30 + 6$$

$$\begin{array}{r} 200 + 30 + 6 \\ 4 \\ \hline 800 + 120 + 24 = 944 \end{array}$$

2. Multiply $a + 4\,b - 2\,a^2c$ by $3\,a$.

$$\begin{array}{l} a + 4\,b - 2\,a^2c \\ 3\,a \\ \hline 3\,a^2 + 12\,ab - 6\,a^3c \end{array}$$

It is evident that the product of the whole multiplicand is equal to the sum of its parts multiplied by the multiplier.

RULE. *To multiply a polynomial by a monomial, multiply each term of the multiplicand by the multiplier as in case of monomials.*

(The multiplier is generally written under the first term, because we begin to multiply at that end. In arithmetic we begin at the right.)

Multiply:

1.
$$\begin{array}{l} 4a + 6c \\ 3a \\ \hline \end{array}$$

2.
$$\begin{array}{l} 7c - 2x^2 \\ -3c^2 \\ \hline \end{array}$$

3.
$$\begin{array}{l} 3ac + 4ab + 9ax \\ -2d^2c^2 \\ \hline \end{array}$$

4.
$$\begin{array}{l} x^2 - 2xy + y^2 \\ -2xy \\ \hline \end{array}$$

5.
$$\begin{array}{l} a^2 - 3ab + b^2 \\ -3ab^2 \\ \hline \end{array}$$

6.
$$\begin{array}{l} ax - ay - az \\ -axyz \\ \hline \end{array}$$

7.
$$\begin{array}{l} ab - cd - ef \\ m \\ \hline \end{array}$$

8.
$$\begin{array}{l} x - y - z \\ -y \\ \hline \end{array}$$

9.
$$\begin{array}{l} a^2 + 4ab - b^2 \\ -3b^2 \\ \hline \end{array}$$

10.
$$\begin{array}{l} a + b + c \\ -x \\ \hline \end{array}$$

11.
$$\begin{array}{l} 3a^2 + 2ab - c^3 \\ 4a^7 \\ \hline \end{array}$$

12.
$$\begin{array}{l} 3amx + 2amy - 6amz \\ -3amxyz \\ \hline \end{array}$$

[37] Multiply:

13. $3ax - 4ay + 2az$ by $3a$.

14. $4ab - 3c^2 + 4d^3$ by $2ac$.

15. $6a^2b + 3a^3d^2 + bd$ by $-3a^2b^2d^2$.

16. $6a^2x^2y^2 + 3b^2xy + 2xy$ by $4x^2y^2$.

17. $3an - 2an^2 - 4am$ by $-3mn$.

18. $4xy - 2my - 3a - 4ab$ by $7abx$.

19. $5\,ac^2 - 3\,a^2c + 6\,a^2c^2$ by $-2\,a^2c^2$.

20. $3\,x - 3\,x^2 - 3\,x^3 - 3\,x^4$ by $3\,x^3$.

21. $5\,ax + 3\,a^2y + 4\,az - 3$ by $+2\,a^2z$.

22. $-ax - 2\,a^2x^2y - 4\,a^2b$ by $-8\,abx$.

23. $a^2b^2 + a^2c^2 + a^2d^2 - a^2e^2$ by $-ae^2$.

ORIGINAL WORK

[38] Make up five examples under this case and find the product.

OPTIONAL WORK

Multiply: 1. $3\,a^2x + 4\,a^2y - 2\,a^2z$ by $2\,a$.

2. $5\,ay - 2\,a^2y^2$ by $3\,a^2y^2z^2$.

3. $3(a+b) + 2(x+y)$ by $3\,a$.

4. $4\,a^{\frac{1}{2}} + 2\,a^{\frac{3}{2}} - a^{\frac{5}{2}}$ by $2\,a^{\frac{1}{2}}$.

5. $\frac{1}{3}\,c^2 - \frac{1}{4}\,c - \frac{1}{2}$ by $\frac{1}{3}\,c$.

6. $a^{2m} + a^m + a^2$ by a^m.

7. $a^m - a^2$ by a^2.

8. $y^n - y^{n+1}$ by y^{n-1}.

9. $2\,a^{m-1} + 3\,a^{m-2} - 5\,a^{m-3}$ by $2\,a^3$.

10. $3(x+y)^2 - 2(x+y)$ by $2(x+y)^2$.

PROBLEMS

1. The daily wages of a workman are a dollars. How much will 10 men earn in 4 days at the same rate?

2. A man bought a horses from a dealer. If each horse cost him c dollars, how much did all cost?

3. If b mechanics can do a job in 6 days, how long will it take one man to do it?

4. A man has a sheep and b times as many cows. How many animals has he?

5. How far can a boy ride on his bicycle in b hours, if he goes $a + 4$ miles an hour?

DIVISION

[39] Division in algebra, like division in arithmetic, is the converse of multiplication and the same principles apply in algebra as in arithmetic.

Division is the process of finding a quantity called a quotient, by which the divisor must be multiplied to produce the dividend.

That is,

$$\textit{Divisor} \times \textit{quotient} = \textit{dividend},$$

or

$$\textit{Dividend} \div \textit{divisor} = \textit{quotient},$$

or

$$\textit{Dividend} \div \textit{quotient} = \textit{divisor}.$$

In algebra, division is generally expressed in the form of a fraction; as $\dfrac{\text{dividend}}{\text{divisor}} = \text{quotient}.$

As $\frac{8}{3}$ means $8 \div 3$, so in algebra $\frac{a}{b}$ means $a \div b$.

Be sure to read this expression "a divided by b," not "a over b." The thought to be expressed is the division thought, which is lost sight of in the latter loose expression.

$\dfrac{a+b}{c+d}$ is read, "the quantity $a+b$ divided by the quantity $c+d$," or "the sum of a and b divided by the sum of c and d."

I. Divide $+20$ by $+5$; $+ab$ by $+a$.

Since $+4$ multiplied by $+5 = +20$ (I, page 40), $\frac{+20}{+5} = +4$. Similarly, $\frac{+ab}{+a} = +b$.

When a positive quantity is divided by a positive quantity, the quotient is positive.

II. Divide -20 by $+5$; $-ab$ by $+a$.

Since -4 multiplied by $+5 = -20$ (II, page 40), $\frac{-20}{+5} = -4$. Similarly, $\frac{-ab}{+a} = -b$.

When a negative quantity is divided by a positive quantity, the quotient is negative.

III. Divide -20 by -5; $-ab$ by $-a$.

Since $+4$ multiplied by $-5 = -20$ (II, page 40), $\frac{-20}{-5} = +4$. Similarly, $\frac{-ab}{-a} = +b$.

When a negative quantity is divided by a negative quantity, the quotient is positive.

IV. Divide $+20$ by -5; $+ab$ by $-a$.

Since $-4 \times -5 = +20$ (III, page 41), $\frac{+20}{-5} = -4$. Similarly, $\frac{+ab}{-a} = -b$.

When a positive quantity is divided by a negative quantity, the quotient is negative.

From these examples we see that

The quotient of like signs is plus and of unlike signs is minus.

[40] To divide a monomial by a monomial.

Divide 45 by 9; 60 by -6; -40 by 10; 25 by -5; -80 by -10; -24 by 4.

Divide $-56x$ by 7.

SOLUTION: $-56x \div 7 = -8x$.

Divide $24a$ by -3; $-60ab$ by $-5a$; $12ab$ by $5a$; $-48x$ by $12x$; $72by$ by -8; $-20xy$ by $4xy$.

Divide a^3 by a; b^5 by b^2; x^7 by x^4.

Since $a^2 \times a = a^3$ (page 42), $\frac{a^3}{a} = a^2$.

Since $b^3 \times b^2 = b^5$ (page 42), $\frac{b^5}{b^2} = b^3$.

Since $x^3 \times x^4 = x^7$ (page 42), $\frac{x^7}{x^4} = x^3$.

Since the exponent in the product equals the *sum* of the exponents in the multiplicand and multiplier, *the exponent of the quotient must equal the exponent in the dividend minus the exponent in the divisor.*

Divide $56a^3b^2d^4$ by $-7ab^2d$.

That is, what quantity multiplied by $-7ab^2d$ will produce $56a^3b^2d^4$.

7 times $8 = 56$.	b^2 times $1 = b^2$.
a times $a^2 = a^3$.	d times $d^3 = d^4$.

Or, $7ab^2d \times 8a^2d^3 = 56a^3b^2d^4$.

As the dividend and divisor have unlike signs, the sign of the quotient must be $-$.

$$-8a^2d^3 \times -7ab^2d = 56a^3b^2d^4;\ \text{or, } \frac{56a^3b^2d^4}{-7ab^2d} = -8a^2d^3.$$

RULE. *To divide one monomial by another, divide the coefficient of the dividend by the coefficient of the divisor for the coefficient of the quotient. To this quotient annex all the different letters, giving each letter an exponent found by subtracting the exponent of the letter in the divisor from the exponent of the same letter in the dividend. If the signs of dividend and divisor are alike, the quotient is* $+$, *if unlike, the quotient is* $-$.

Divide:

1. $24a^2$ by $6a$. **2.** $18a^2y$ by $3a$. **3.** $-27a^2x^2y$ by 9.

Divide:

4. $\dfrac{24a^2}{6a}$. **8.** $\dfrac{18a^2y}{-3a}$. **12.** $\dfrac{-27a^2x^2y^2}{9xy}$.

5. $\dfrac{-120a^7y^8}{-6a^2y^3}$. **9.** $\dfrac{72a^4m^7z^4}{9a^3m^3}$. **13.** $\dfrac{36m^{10}}{-3m}$.

6. $\dfrac{8a^2y}{-a^2y}$. **10.** $\dfrac{-44m^4n^3}{11mn}$. **14.** $\dfrac{39d^4e^3}{-3d^2e}$.

7. $\dfrac{13a^4x^2y}{a^2x^2}$. **11.** $\dfrac{-30abcd}{-3ad}$. **15.** $\dfrac{64a^4b^4c^4}{16abc}$.

ORIGINAL WORK

[**41**] Make up three examples like these and find the quotients. Prove.

When the dividend is a polynomial.

Divide $4a^2b + 6ab^2 - 2ab^3$ by $2ab$.

The question is to find the algebraic expression which, when multiplied by $2ab$, will give $4a^2b + 6ab^2 - 2ab^3$.

Now we have learned that $2a$ multiplied by $2ab$ will give us the first term ($4a^2b$) of the dividend, and $3b$ multiplied by $2ab$ will give the second term ($6ab^2$), and $-b^2$ multiplied by $2ab$ will give the third term ($-2ab^3$).

Hence $2a + 3b - b^2$ multiplied by $2ab$ will give $4a^2b + 6ab^2 - 2ab^3$.

$2a + 3b - b^2$ is therefore the quotient of $4a^2b + 6ab^2 - 2ab^3$ divided by $2ab$.

RULE. *To divide a polynomial by a monomial, divide each term of the dividend by the divisor as in the case of monomials.*

Divide:

1. $6ay - 3a^2y^2 + 9ay^3$ by $3ay$.
2. $10x^2y^3 - 15x^2y^2z - 5x^2y^5z^3$ by $5x^2y^2$.
3. $27a^3bc^2 - 9a^3b^2 - 6a^4b^2x^2$ by $3a^3b$.
4. $8a^3b^2 - 4ab^4 - 8b$ by $4b$.
5. $3a^2b - 2a^2b^2 - a^2$ by a^2.

[42] 6. $4a^2x^2 + 3ax + 7a^2xy$ by $+ax$.

7. $7a^2 - 14ax + 21xy - 7x^2$ by 7.
8. $4a^2y^2 + 8a^3y - 6ax^2y + 8amy + 4ay$ by $2ay$.
9. $9abc + 12a^2c^2 - 6a^3c^2x + 24ac$ by $3ac$.
10. $16m^2n^2 - 8mn + 48m^3n^2y - 32m^2n$ by $8mn$.
11. $4an - 24a^3n^2y - 4acn - 28a^4n^2$ by $4an$.
12. $14a^2x^2y^2 - 21a^2x^3y^3z + 56a^3x^3y^3$ by $7a^2x^2y^2$.

13. $720\,a^5x^7 - 480\,a^2x^5 + 640\,a^3x^4$ by $80\,a^2x^4$.

14. $3\,a^2x + 7\,a^3x^2 + 3\,a^2x^3y^2 - 2\,axy$ by $-ax$.

15. $4(a-y) + 6(a-y)^2 + 12(a-y)^3$ by $2(a-y)$.

OPTIONAL WORK

Divide:

1. $8\,a^5 + 16\,a^7$ by $-8\,a^2$.

2. $-x^4y - x^3yz^2 - x^2y^2z^3 + xy^2z$ by $-xy$.

3. $3(x+y) - 6(x+y)^2 + 9(x+y)^3$ by $3(x+y)$.

4. $2\,a^2 + 8\,a^3 + 12\,a^4$ by $2\,a^2$.

5. $(1-a) - (1-a)^2$ by $(1-a)$.

6. $a(c-d) - b(c-d)$ by $(c-d)$.

7. $25\,a^2b^3x - 15\,abx^2 - 5\,ab^2x$ by $5\,ab^2x$.

THE PARENTHESIS

[43] You have learned that the parenthesis is used to inclose quantities which are to be treated as one quantity, and that the sign before it affects the whole expression inside the parenthesis.

You also learned that you can remove a parenthesis preceded by a minus sign if you change the signs of all the quantities within the parenthesis. Likewise, if you wish to inclose quantities within a parenthesis preceded by a minus sign, the signs of the quantities must be changed. This is necessary in order to produce the original expression when the parenthesis is removed.

If the quantity within the parenthesis has a coefficient, the whole expression must be multiplied by it.

Remove the parenthesis from the following expressions:

1. $2a-(a+b)$.
2. $7a-8b-2(3a-2b)$.
3. $3a-8x-2(2a-4x)$.
4. $3x+7y-3(x-2y+4)$.
5. $9a-x-2(2a-x-3)$.
6. $6b-4a+4-2(b-2a+2)$.
7. $3a-4x-(a-3x)+(x-4a)$.
8. $x-y+z-(x+y-z)-(x+y+z)-(y+z-x)$.

9. Write three algebraic expressions, each having a parenthesis with the minus sign and remove the ( ).

10. Place the last two terms of the expression $a+b-c+d$ within a parenthesis with the minus sign before it.

11. Insert the second and third terms within a parenthesis with the minus sign before it.

12. Place the second and fourth terms within a parenthesis with the minus sign before it.

13. Inclose the second and third terms of $3ab-2a^2+3b-12ac+10$ within a parenthesis with the minus sign.

14. Inclose the second, third, and fifth terms of the above expression within a parenthesis with a minus sign.

15. Write a polynomial of four terms. Introduce the last three terms within a parenthesis with the minus sign before it.

FACTORING

[44] The **factors** of a quantity are the quantities which, when multiplied together, will produce it.

To factor a monomial.

1. What are the factors of 16? of 8? of 10?

2. What are the factors of $7a^3b^2c$?

They are readily seen to be 7, a, a, a, b, b, and c, as these quantities multiplied together will produce $7a^3b^2c$.

Factor:

1. $15a^2b^2$.	4. $64m^2n^2$.	7. $144x^2y^2z^2$.
2. $12x^3y^2$.	5. $33a^3b^2$.	8. $16a^4b^4c^4$.
3. $48a^2b^3$.	6. $25m^6n^5$.	9. $27a^3b^3c^3$.

[45] To find two factors of a polynomial, one of which is a monomial.

Factor $6a^2b^3 + 3a^2bc - 9a^3b^2d$.

We desire to find the largest monomial which will divide all the terms. We find that 3 is the largest number which will divide the coefficients, also that a^2 and b will divide all the terms, and that no other letter is common to them all.

Hence $3a^2b$ is common to all the terms. Divide the expression by $3a^2b$ to find the other factor.

$3a^2b$ and $2b^2 + c - 3bd$ are the factors required. They may be written as $3a^2b(2b^2 + c - 3bd)$; that is, the monomial factor may be written as a coefficient to a polynomial factor placed in a parenthesis.

RULE. *Divide the polynomial by the highest divisor common to all the terms. The divisor and the quotient will be the factors required.*

Find two factors of each of the following algebraic expressions, one of which is a monomial:

1. $4a - 4b$.	3. $a^3b + 4a^3c$.	5. $32xy - 48x^2y^2z$.
2. $a^2 - ab$.	4. $7a^2 + 14ab$.	6. $72x^2y^2 - 54xy$.

[46]

7. $9a^2b^2 - 18a^3b^2$.	15. $14a^2b - 7a^3b^2x - 21a^4b^3$.
8. $8a^3xy - 64x^2z$.	16. $32axy - 16a^2b^2 + 24a^3bx$.
9. $28abc - 42ayz$.	17. $40x^2y - 60x^3y^3 - 80x^4y^4$.
10. $7a^2b^2 + 3a^2bx$.	18. $9m^2n^2 - 27mnx$.
11. $3a^3xy + 9ax$.	19. $13am - 39am^2$.
12. $4b^2xy^2 + 18aby^2$.	20. $24a^3x^2 + 42a^2x^3y$.
13. $3xy^2 + 27xy^3 - 3xy^4$.	21. $3ay - 6ay^2 + 12$.
14. $2x^2y^2 - 4x^2y^2z - 8ax^2y^2$.	22. $4ab - 8a^2n^2 + 8anx$.

[47]

23. $6abc - 12a^2b^2 - 18a^3bc$.	26. $16a^3x^2y - 20a^2x^2y^2$.
24. $7x^2y^2 - 7x^2yz^2 - 7xyz$.	27. $25m^2x^3y^7 - 60m^3xy^3$.
25. $10ab^2m - 5a^2bm + 15ab^2m^2$.	28. $49a^2x^7y^3 + 63a^2x^3y^4$.

ORIGINAL WORK

Make up three examples like the above, and factor them.

OPTIONAL WORK

Factor:
1. $9m^2n^2 - 27mny + 81m^3n^3$.
2. $a^2b^2x^2 - 3a^3b^3x^3 - 6a^4b^4x^4$.
3. $2a^mb + 4a^{2m}c - 8a^{2m}d$.
4. $4x^{\frac{1}{2}}y^2z - 16x^{\frac{3}{2}}y^4 - 28x^{\frac{1}{2}}y^3$.
5. $32a^nb^mc^2 - 64a^{2n}b^mc^3 - 16a^nb^m$.

LOWEST COMMON MULTIPLE

[48] Name three numbers that will contain 4; three that will contain 11.

Name three quantities that will contain a; three that will contain ax.

What quantity or number will exactly contain 2, 6, and 8? 7, 4, and 2? 6, 12, and 24?

What quantity will contain 2, 4, and a? 3, 4, a, and a^2? $2a$, $6ax$, and $12a^2x^2$?

Your answers have not necessarily been the *lowest* quantities that will contain the quantities given. Rewrite these examples, giving the lowest quantity in each set that can be divided by each quantity.

What is the lowest quantity that will contain a, a^2, a^4, and a^7?

In arithmetic the term "least common multiple" is correct, as it is the least or smallest number that will contain each of the numbers; but in algebra, where we deal with literal quantities, we should say "**lowest common multiple,**" as we mean the quantity having the lowest exponent or degree.

A **multiple** of a quantity is one or more times that quantity.

A **common multiple** of two or more quantities is any quantity that will contain each of them.

The **lowest common multiple** of two or more quantities is the quantity of the lowest degree that will contain each of them.

The lowest common multiple of two or more quantities must contain all the factors of each quantity, and no others.

Find the L. C. M. of $2ax$, $6a^3y$, and $8a^2xy^2$.

The factors of the quantities are:

$$2ax = 2 \times a \times x.$$
$$6a^3y = 2 \times 3 \times a \times a \times a \times y.$$
$$8a^2xy^2 = 2 \times 2 \times 2 \times a \times a \times x \times y \times y.$$

The L. C. M. must contain the greatest number of the different factors found in any quantity. It will, therefore, contain:

Three 2's, One 3, Three a's, One x, Two y's,

or $2 \times 2 \times 2 \times 3 \times a \times a \times a \times x \times y \times y$, or $24a^3xy^2$.

This is the quantity of the lowest degree that each quantity will divide.

The L. C. M. of monomials is generally found by inspection.

Find by inspection the L. C. M. of $2ax$, $6a^3y$, and $8a^2xy^2$.

It is readily seen that each letter with its highest exponent must be used. This will be a^3xy^2.

If to this we prefix as coefficient the L. C. M. of the numerical coefficients, we shall have the L. C. M. required, or $24a^3xy^2$.

RULE. *To find the L. C. M. of monomials, write the different letters found in the monomials, each with the highest exponent found in any one. To this prefix as coefficient the least common multiple of the numerical coefficients.*

Find the lowest common multiple (L. C. M.) of:

1. $4a^2b$, $8a^3b^2$, and $12abx^2$.
2. $3abc$, $9a^3bc^2$, and $12a^2b^3c^3$.
3. $6a^2b^2x$, $4a^3bx^2y$, and $3ab^2xy^2$.
4. $4x^2y^2$, $9xy^2z$, and $12x^3y^3z^3$.
5. $5a^2c^2x$, $2acx^2$, $4a^2cx$, and $10a^2cx^3$.
6. $2bc^2x^2$, $7ab^2c^2x$, $4a^2b^3c^2x^2$, and $4abcx$.

ORIGINAL WORK

Make up three examples in L. C. M.

OPTIONAL WORK

Find the L. C. M. of each of the following quantities:

1. $2m^2n^2p$, $8m^4n^2$, $5m^2n^3p$, and mnp.
2. $6x^3y^2z$, $12xy^3z^4$, and $15x^4y^4z^4$.
3. $9a^2bc^2$, $12a^2c$, and $18b^2cd^2$.
4. $16a^2bd^2$, $32a^3bx$, and $8x^2y^5$.
5. $3a^2c$, $9a^3c^4$, and $36abcd$.
6. $8x^3y$, $48axy$, and $18a^2x^2y^2$.
7. $15a^2b^3$, $45abc^2$, and $18a^4bc^3$.
8. $32b^2c^2$, $64x^2y^2$, and $48b^2y^2$.
9. $10x^3y$, $90x^4y^2$, and $45xy^5$.
10. $7b^2x$, $12a^2x^2$, and $21c^2x^3$.

FRACTIONS

[**49**] You have learned in arithmetic that a fraction is an expression meaning one or more equal parts of a unit. It is also an expression of division. Thus $\frac{3}{4}$ may mean 3 of the 4 equal parts into which a unit is divided. It may also mean $\frac{1}{4}$ of 3 units. $\frac{1}{4}$ of three apples may be equal in amount to 3 of the 4 equal parts into which one apple is divided; but the process of reasoning is entirely different.

In algebra a **fraction** is an expression of division.

The **numerator** is the dividend and the **denominator** is the divisor.

All the principles of division given on page 47 may be applied to fractions. This knowledge will help you to understand the relation of the signs of the terms.

Thus, $8 \div 4 = \frac{8}{4} = 2.$ $2\,ax \div a = \frac{2\,ax}{a} = 2\,x.$

A fraction has three **signs**; one before the fraction, one before the numerator, and one before the denominator.

Thus, $\frac{2\,ab}{b}$ means $+\frac{+2\,ab}{+b}$.

As in division, the result is not changed if we change the sign in both the numerator and the denominator; as,

$\frac{2\,ab}{b} = 2\,a.$ $\frac{-2\,ab}{-b} = 2\,a.$ Why are these results alike?

If one sign is changed, the result is changed in sign. Thus,

$\frac{2\,ab}{b} = 2\,a.$ But $\frac{-2\,ab}{b} = -2\,a$ and $\frac{2\,ab}{-b} = -2\,a.$ (Why?)

If all three signs are changed, the sign of the result is changed. Thus,

$\frac{2\,ab}{b} = 2\,a,$ $\frac{-2\,ab}{-b} = 2\,a.$ But $-\frac{-2\,ab}{-b} = -2\,a.$ (Why?)

If any two signs are changed, the result is unchanged.

Thus, $\frac{2\,ab}{b} = 2\,a,$ $\frac{-2\,ab}{-b} = 2\,a,$ $-\frac{-2\,ab}{b} = -(-2\,a)$

$= 2\,a,$ $-\frac{2\,ab}{-b} = -(-2\,a) = 2\,a.$ (Why?)

Likewise, if we change any two sets of signs of the fraction, the value will remain unchanged. Thus,

$$\frac{a-b}{c}=\frac{b-a}{-c}=-\frac{b-a}{c}=-\frac{a-b}{-c}.$$

Change any two of the three signs in each of the following fractions and show that the value is unchanged:

$$\frac{18}{3},\ \frac{-ab}{b},\ \frac{3c}{c},\ \frac{15-5}{2},\ \frac{a-b}{d}.$$

An **entire quantity** is one in which no part is a fraction.

A **mixed quantity** is one that consists of a whole quantity and a fraction.

A fraction is in its **lowest terms** when the numerator and denominator have no factors in common; or when, as we say in arithmetic, they are prime to each other.

REDUCTION OF FRACTIONS

[50] To reduce a fraction to its lowest terms.

Reduce $\frac{120}{144}$ to lower terms.

$$\frac{2)120}{2)144}=\frac{2)60}{2)72}=\frac{2)30}{2)36}=\frac{3)15}{3)18}=\frac{5}{6}.$$

Reduce $\frac{9a^2b^3x}{12a^4b^2xy}$ to lowest terms.

As in arithmetic, we cancel the factors common to both the numerator and denominator.

We see that 3, a^2, b^2, and x will divide each term. Hence,

$$\frac{3a^2b^2x)9a^2b^3x}{3a^2b^2x)12a^4b^2xy}=\frac{3b}{4a^2y}.$$

This is evidently in its lowest terms, and the value is unchanged, as we have divided the dividend and the divisor by the same quantity.

RULE. *To reduce a fraction to its lowest terms, cancel from both numerator and denominator all the common factors.*

Reduce the following fractions to their lowest terms:

1. $\frac{4a^2b^2}{8ab^2x}$.
2. $\frac{3a^3b^2y}{27a^2b^3x}$.
3. $\frac{12a^2b^2x^2}{9a^3b^3xy}$.
4. $\frac{18a^2b^4c^2}{27a^3bc^4d}$.
5. $\frac{20a^5c^6d}{25abc^6d^2}$.
6. $\frac{12x^2y^3z}{15x^3y^4z^2}$.
7. $\frac{16abc}{24a^2b^2c^2}$.
8. $\frac{25a^5x^2z^7}{125a^6x^7z^2}$.
9. $\frac{26a^2y^4z}{39a^4y^4z^2}$.
10. $\frac{144a^2b^4cx^2}{72a^3cx^2y}$.
11. $\frac{63x^3y^2z}{119abxy^3}$.
12. $\frac{182a^3b^4c^2}{221a^7b^3cm}$.

Reduce $\frac{2ax^2+4a^2x}{4a^2xy}$ to its lowest terms.

In this and like examples we first find the factors of the numerator and then reduce by cancellation.

Thus, $$\frac{2ax^2+4a^2x}{4a^2xy}=\frac{2ax(x+2a)}{4a^2xy}=\frac{x+2a}{2ay}.$$

[51] Reduce the following to their lowest terms:

1. $\frac{4x^2-6ax}{2xy}$.
2. $\frac{6a^2-9ab}{3ab}$.
3. $\frac{14ax^2-7ax}{7x}$.
4. $\frac{5a^2b-10b^2}{5ab^2}$.

5. $\dfrac{3\,ax - 9\,a^2x^2}{3\,axy}$.

6. $\dfrac{8\,abc - 12\,a^2c}{4\,cx}$.

7. $\dfrac{3\,a^2cd - 3\,ac^2d}{3\,cd}$.

8. $\dfrac{25\,ac^2y - 10\,ac^2x}{5\,cxy}$.

9. $\dfrac{4\,a^2b^2d - 6\,a^2b^2c}{2\,a^2b^2c}$.

10. $\dfrac{3\,a^3c^3x - 3\,a^2bc^3x}{3\,a^2c^3x}$.

RIGINAL WORK

Write two fractions and reduce them to their lowest terms.

OPTIONAL WORK

Reduce to their lowest terms:

1. $\dfrac{24\,a^3x^2y^3}{60\,a^5y^4}$.

2. $\dfrac{16\,x^3y^7z}{18\,x^5y^2}$.

3. $\dfrac{12\,a^5z^2}{144\,a^5z^3}$.

4. $\dfrac{36\,a^7xy^m}{90\,a^6xy^m}$.

5. $\dfrac{50\,b^{2m}c^n}{90\,b^mc^{2n}}$.

6. $\dfrac{27\,x^{3m}y^2}{72\,ax^{2m}y^2}$.

7. $\dfrac{44\,a^{\frac{3}{2}}b^5c}{77a^{\frac{5}{2}}b^7}$.

8. $\dfrac{6\,a^2c - 8\,a^2d}{6\,a^3}$.

9. $\dfrac{12\,ac + 9\,acd}{9\,ac}$.

10. $\dfrac{3(a+b)^2}{6(a+b)}$.

11. $\dfrac{14(x-y)^3}{21(x-y)}$.

12. $\dfrac{3\,ac^m - 6\,ad^n}{3\,a}$.

13. $\dfrac{3\,x(a+b)^2}{9\,x^2(a+b)}$.

14. $\dfrac{39\,a^3(x+y)}{13\,a^2(x+y)^2}$.

[52] To reduce a fraction to an entire or a mixed quantity.

Reduce $\frac{4a^2b - 6a^2b^2}{2a^2b}$ to an entire or a mixed quantity.

Since a fraction is an expression of division, if we perform the division required, we shall have

$$(4a^2b - 6a^2b^2) \div 2a^2b = 2 - 3b,$$

an entire quantity.

Reduce $\frac{3abx - 2axy - d}{ax}$ to a mixed quantity.

Perform the division indicated as far as possible. Then place the remainder as a numerator and the divisor as a denominator, and connect it to the entire part with the proper sign.

$$(3abx - 2axy - d) \div ax = 3b - 2y - \frac{d}{ax}.$$

The first two terms are divisible by the denominator, but the third term is not. Express this undivided part as the numerator of a fraction of which the divisor is the denominator.

Reduce the following fractions to entire or mixed quantities:

(Retain the answers to these examples.)

1. $\frac{x^2+1}{x}$.
2. $\frac{3x^2 - x + 3}{x}$.
3. $\frac{x^3 + x^2 + x + 1}{x}$.
4. $\frac{3ax + 2b}{ax}$.
5. $\frac{25ab + 3c}{5}$.
6. $\frac{a^3 + b^3}{a^2}$.
7. $\frac{14a^2x - 8ax^2}{2ax}$.
8. $\frac{20x^2 - 40x + 3}{5x}$.

(Be careful of the signs of the fractional part.)

9. $\dfrac{6xy + 2abx + 2x}{2x}$.

10. $\dfrac{9x^2 - 18x - 7}{3x}$.

11. $\dfrac{16x^3 - 8x - 5}{8x}$.

12. $\dfrac{4a^4 + 16a^2 + 3}{4a^2}$.

13. $\dfrac{63a^2x - 54ax^2 - 8a}{9ax}$.

14. $\dfrac{33x^2y^2 + 44x^3y^3 - 55xy}{11x^2y^2}$.

ORIGINAL WORK

Make up two or more examples under this case.

OPTIONAL WORK

Reduce the following fractions to mixed quantities:

1. $\dfrac{7a^2 - 14ab + 1}{7a}$.

2. $\dfrac{3ac - 9a^2c^2 - x}{3ac}$.

3. $\dfrac{2a^mb - 4a^{2m}b^2 - c}{2a^mb}$.

4. $\dfrac{9x^2y - 27x^2y^2 - 33ax^2y - 15a^2x^2y^2}{3x^2y}$.

5. $\dfrac{13xy - 39x^2y^2 + 52x^3y^3 - 13a}{13xy}$.

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[53] To reduce a mixed quantity to a fractional form.

Reduce $4\frac{2}{5}$ to a fraction.

$$4\tfrac{2}{5} \text{ means } 4 + \frac{2}{5} = \frac{4 \times 5}{5} + \frac{2}{5} = \frac{4 \times 5 + 2}{5} = \frac{20 + 2}{5} = \frac{22}{5}.$$

In algebra the same principles apply.

Reduce $3x + \frac{4b}{c}$ to a fractional form.

$$3x = \frac{3x \times c}{c} = \frac{3cx}{c}. \qquad \frac{3cx}{c} + \frac{4b}{c} = \frac{3cx + 4b}{c}.$$

RULE. *Multiply the whole quantity by the denominator, add the numerator, and write the sum over the denominator.*

Reduce the following mixed quantities to fractions:

1. $2x + \frac{5x}{9}$.
2. $3ab + \frac{5ab}{7}$.
3. $ay + \frac{6ay}{9}$.
4. $4a^2 - \frac{3a^2}{5}$.
5. $a^2 - \frac{b^2}{a}$.
6. $x - \frac{y^2}{x}$.
7. $a^2 - \frac{b^2}{a^2}$.
8. $3a^2 - 4a + \frac{5b}{2}$.
9. $4a + \frac{3x+5}{2a}$.
10. $x + \frac{3ax - a^2}{x}$.
11. $8b + \frac{4x+c}{3}$.
12. $2m + \frac{2n+n^2}{5x}$.
13. $\frac{2a-4c}{7} - 3a + 5c$.
14. $\frac{ay-4y}{6y} - 2a + 3$.

ORIGINAL WORK

Make up two or more examples of mixed quantities, and reduce each to a fractional form.

OPTIONAL WORK

In the previous case your answers are mixed quantities. Reduce each of them to a fractional form and see whether it is the same as the fraction given in the example.

ADDITION AND SUBTRACTION OF FRACTIONS

[54] As in arithmetic, only fractions having a common denominator can be added or subtracted.

Thus, $\frac{3}{4}+\frac{5}{6}+\frac{7}{8}=\frac{18}{24}+\frac{20}{24}+\frac{21}{24}=\frac{59}{24}=2\frac{11}{24}$.

The same process is pursued in algebra.

RULE. *To add fractions, reduce them, if necessary, to a common denominator, add the numerators for a new numerator, and place the sum over the common denominator. Reduce the resulting fraction to its lowest terms.*

RULE. *To subtract fractions, reduce them, if necessary, to a common denominator, subtract the numerator of the subtrahend from the numerator of the minuend, and write the difference over the common denominator.*

As in arithmetic, if the quantities are mixed quantities, the entire parts may be added or subtracted separately without being reduced to fractional forms.

Add $\frac{2a}{b}$, $\frac{3c}{2b^2}$, and $\frac{4c}{3ab}$.

The L. C. M. of the denominator is $6ab^2$. (Why?)

Multiply both terms of the first fraction by $6ab$, both terms of the second fraction by $3a$, and both terms of the third by $2b$. The values of the fractions will be unchanged (why?), and we shall have

$$\frac{12a^2b}{6ab^2}+\frac{9ac}{6ab^2}+\frac{8bc}{6ab^2}.$$

These fractions have now a common denominator, and the sum will be

$$\frac{12a^2b+9ac+8bc}{6ab^2}.$$

Add:

1. $\dfrac{2x}{4}, \dfrac{3x}{8}, \dfrac{5x}{12}.$

2. $\dfrac{2a^2}{3}, \dfrac{4a^2}{15}, \dfrac{7a^2}{9}.$

3. $\dfrac{a^2}{b}, \dfrac{3a^2}{2b}, \dfrac{5a^2}{4b}.$

4. $\dfrac{2}{3b^2}, \dfrac{3}{2b^2}, \dfrac{5}{6b^2}.$

5. $\dfrac{a+y}{4}$ and $\dfrac{a-y}{3}.$

6. $\dfrac{a+b}{9}, \dfrac{a-b}{15}, \dfrac{b-a}{3}.$

7. $\dfrac{a+8}{3}, \dfrac{a+7}{4}, \dfrac{a-5}{2}.$

8. $3a+\dfrac{4y}{b}$ and $2a-\dfrac{3x}{b^2}.$

(Add the entire parts of the expression and the fractional parts separately.)

[55] 9. $x-3+\dfrac{3x+4}{4x}$ and $4x+3+\dfrac{2x-1}{2x}.$

10. $3a+4y+\dfrac{6a-4}{3x}$ and $3y-3a+\dfrac{3-2ax}{x^2}.$

11. $\dfrac{ac}{b}, \dfrac{bc}{a},$ and $\dfrac{ab}{c}.$

12. $\dfrac{a+x}{4}, \dfrac{a-x}{8},$ and $\dfrac{x-a}{12}.$

13. From $2a-4+\dfrac{4a+7}{3a}$ take $6a-5+\dfrac{3a+2}{6a}.$

14. From $9x+17+\dfrac{4x+11}{3x}$ take $4x+9+\dfrac{2x+2}{x}.$

Simplify: 15. $\dfrac{x-y}{4}+\dfrac{x+y}{5}-\dfrac{x}{2}+\dfrac{y}{10}.$

16. $\dfrac{3ax}{4}-\dfrac{3ax}{8}+\dfrac{5ax}{12}.$

17. $\dfrac{a+b}{ab}+\dfrac{b-c}{bc}+\dfrac{a-c}{ac}.$

OPTIONAL WORK

Make up two examples in addition and two in subtraction, and find the answers.

Simplify:

1. $\frac{3x-2}{5}+\frac{3}{4}-\frac{4x+6}{8}-\frac{x-1}{2}.$

2. $\frac{4}{3x^2y}+\frac{2}{6xy^2}-\frac{2}{2x^2y^2}-\frac{2x-1}{6x^2y^2}.$

3. $\frac{x^2+2xy+y^2}{xy}-\frac{x^2-2xy+y^2}{xy}.$

4. $\frac{a}{9}+\frac{a}{6}+\frac{a}{18}+\frac{a}{12}+\frac{a}{3}+\frac{a}{4}.$

5. $\left(6a-\frac{3a}{4}\right)+\left(\frac{a}{5}-8a\right)+\left(5a-\frac{a}{2}\right).$

6. $\frac{a-b}{ab}+\frac{c+a}{ac}-\frac{c-b}{cb}.$

7. $\frac{x-y}{xy}+\frac{z-x}{zx}+\frac{y-z}{yz}.$

MULTIPLICATION OF FRACTIONS

[56] The principles of multiplication in arithmetic apply to multiplication in algebra.

Multiply $\frac{3}{7}$ by 5.

This $= \frac{15}{7}$, since multiplying the numerator, or dividing the denominator, multiplies the fraction.

Multiply 5 by $\frac{3}{7}$.

Since this is the same as $\frac{3}{7} \times 5$, the result is $\frac{15}{7}$.

Multiply $\frac{3}{a}$ by 5.

For the same reason, this is $\frac{15}{a}$.

Show that $5 \times \frac{3}{a} = \frac{15}{a}$.

Multiply $\frac{5}{3a}$ by $6a$; $4b$ by $\frac{2a}{b}$; $\frac{3x}{2y^2}$ by $4y$.

Multiply $\frac{2}{3}$ by $\frac{4}{5}$.

$\frac{2}{3} \times 4 = \frac{8}{3}$. But $\frac{2}{3}$ is to be multiplied by $\frac{1}{5}$ of 4 (not 4), hence the multiplier used is five times too large, and the product is therefore five times too large. To obtain the proper result, we must divide the product by 5. This can be done by multiplying the denominator by 5. The result is $\frac{8}{3 \times 5}$ or $\frac{8}{15}$. That is, $\frac{2}{3} \times \frac{4}{5} = \frac{2 \times 4}{3 \times 5} = \frac{8}{15}$.

Multiply $\frac{a}{b}$ by $\frac{c}{d}$.

$\frac{a}{b} \times c = \frac{ac}{b}$. The result is d times too large (as the multiplier is $\frac{1}{d}$ part of c, not c). Hence, to secure the correct result we multiply the denominator by d, or $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$.

RULE. *To multiply a fraction by an entire quantity, multiply the numerator of the fraction by the whole quantity and divide the result by the denominator, canceling when possible.*

To multiply a fraction by a fraction, multiply the numerators together for a new numerator, and the denominators together for a new denominator, canceling when possible.

Mixed quantities should be expressed in fractional form before multiplying.

DIVISION OF FRACTIONS

The principles of division in algebra are the same as those in arithmetic.

Divide: 1. $\frac{4}{5}$ by 2, $\frac{8}{9}$ by 4, $\frac{72}{121}$ by 8.

2. $\frac{4}{5}$ by 3, $\frac{8}{9}$ by 5, $\frac{7}{15}$ by 4.

3. $\frac{2}{3}$ by $\frac{4}{5}$, $\frac{7}{8}$ by $\frac{5}{6}$, $\frac{5}{42}$ by $\frac{15}{63}$.

4. $\dfrac{3a}{4}$ by 3, $\dfrac{12a}{7}$ by $4a$, $\dfrac{7b}{15}$ by 7.

5. $\dfrac{3a}{2}$ by 4, $\dfrac{5a}{7}$ by 3, $\dfrac{3a}{5}$ by 4.

6. Divide $\frac{2}{3}$ by $\frac{4}{5}$. 7. Divide $\dfrac{a}{b}$ by $\dfrac{c}{d}$.

$\dfrac{a}{b} \div c = \dfrac{a}{bc}$, since multiplying the denominator of a fraction divides the fraction. But we were not to divide the fraction by c, but by $\dfrac{1}{d}$ part of c. Hence the result is d times too small. (Why?) To give the correct quotient, we must multiply the result by d, which gives $\dfrac{ad}{bc}$. This is the same as $\dfrac{a}{b} \times \dfrac{d}{c}$ or $\dfrac{ad}{bc}$. That is, we invert the divisor and multiply.

RULE. *To divide a fraction by an entire quantity, divide the numerator by the entire quantity or multiply the denominator by the quantity.*

To divide an entire quantity by a fraction, or a fraction by a fraction, invert the divisor and proceed as in multiplication.

[57] Perform the operations indicated:

Set I

1. $\dfrac{3ab^2}{4cd} \times 8c^2d^2$.

2. $\dfrac{7ac}{5d^2} \times 5d$.

3. $\dfrac{5\,ab^2c}{13\,xy} \times 39\,x^2y^2.$

4. $\dfrac{5\,x^2y}{27\,ab^2} \times 6\,a^2b.$

5. $\dfrac{16\,a^2x}{9\,b^2c} \times 18\,b^3c^2.$

6. $\dfrac{3\,ab}{4\,c} \div ab.$

7. $\dfrac{7\,xy}{3a} \div 14\,a.$

8. $\dfrac{4\,x^2y}{5\,ab} \div 6\,x.$

9. $\dfrac{9\,ac}{4} \div 8\,a.$

10. $\dfrac{ab}{7\,c} \div 4\,c.$

[58] Set II

1. $\dfrac{3\,ac}{2\,bd} \times \dfrac{4\,b^2}{6\,c}.$

2. $\dfrac{a-b}{c^2} \times \dfrac{c}{b}.$

3. $\dfrac{5\,x^4}{3\,ac} \times \dfrac{12\,a^2c}{15\,x}.$

4. $\dfrac{m-n}{18\,b} \times \dfrac{18\,b}{5\,d}.$

5. $\dfrac{a+b}{3\,c} \times \dfrac{3\,c^2}{2\,d}.$

6. $\dfrac{x}{y} \div \dfrac{x^3}{y^3}.$

7. $\dfrac{4\,ac}{5\,bd} \div \dfrac{2\,a^2c^2}{15\,b^2d^2}.$

8. $\dfrac{8\,a^2b^2}{9\,x^2y^3} \div \dfrac{16\,ab}{27\,x^2y^2}.$

9. $\dfrac{3\,c^2d}{2\,d} \div \dfrac{9\,ac^2}{10\,d^2}.$

10. $\dfrac{a-b}{x} \div \dfrac{2\,a}{3\,x^2}.$

[59] Set III

1. $\dfrac{2\,ab}{3\,c} \times \dfrac{4\,c^2d}{6\,a^2b} \times \dfrac{9\,ab^2}{12\,d^2}.$

2. $\dfrac{4\,bc^2}{5\,ad} \times \dfrac{7\,a^2d^2}{8\,xy} \times \dfrac{10\,x^2y^2}{14\,abc}.$

3. $\dfrac{8\,ay}{3\,x} \times \dfrac{9\,a^3x^2}{16\,y^2} \times \dfrac{4\,y}{6\,a}.$

4. $\dfrac{2\,a^2b}{4\,d^2m} \times \dfrac{8\,d^3m^3}{3\,abx^2} \div \dfrac{16\,a^2b^2}{9\,x^2y^2}.$

5. $\dfrac{15x^2}{xyz} \times \dfrac{18z^3}{40z} \div \dfrac{81z^3}{27x^2}.$

6. $\dfrac{a^2n}{b^2m} \times \dfrac{4an}{3bm^2} \div \dfrac{8a^3n}{9b^2m^3}.$

7. $\dfrac{(a-x)}{3b} \times \dfrac{4a^2b^2}{2(a-x)} \div \dfrac{8a^3b^4}{2x^3}.$

8. $\left(\dfrac{2a}{3} + \dfrac{3a}{6}\right) \div \left(\dfrac{4c}{6} - \dfrac{5c}{9}\right).$

9. $\dfrac{2a^2}{7cy} \times \dfrac{3d}{4a^3} \div \dfrac{21c^2y^2}{9ad^2}.$

10. $\dfrac{3ab}{16xy} \times \dfrac{4cd}{7mn} \times \dfrac{28m^2x^2}{9a^2d^2}.$

OPTIONAL WORK

1. $\dfrac{a}{bc} \times \dfrac{a}{c(a-b)}.$

2. $\left(a + \dfrac{y}{x}\right) \times \dfrac{x}{a}.$

3. $\dfrac{a-x}{c} \div \dfrac{a-x}{b}.$

4. $\dfrac{ac+ad}{16} \times \dfrac{4}{ac}.$

5. $\dfrac{2ax+6ay}{3bc} \div \dfrac{2a}{9b}.$

6. $\dfrac{a^m}{y^m} \times \dfrac{a^m}{y^m}.$

7. $\dfrac{x^n}{y^n} \div \dfrac{y^m}{x^m}.$

8. $\left(x - \dfrac{1}{x}\right) \div \left(\dfrac{x^2-1}{2x}\right).$

9. $\dfrac{ax+ay}{9bx} \times \dfrac{3b^2x^2}{2ax}.$

10. $\dfrac{(a+b)^2}{8a^2b^2} \div \dfrac{(a+b)}{4ab}.$

11. $\dfrac{(x-y)^2}{4a^3b^2} \div \dfrac{x-y}{2ab}.$

12. $a - \dfrac{1}{a} \div \dfrac{a^2-1}{2a}.$

EQUATIONS

[60] You have been dealing, heretofore, mainly with what has been called "literal arithmetic"; that is, the general application of those rules and operations with which you are familiar in arithmetic. Equations were introduced on pages 13 to 15, and you have seen how simple problems were easily solved by them. We shall now take up more carefully the subject of the equation.

An **equation** is an expression of equality between two quantities or sets of quantities.

Thus, $3+6=9$ is an equation. $x=4$, $a=b$, $2a+x=3b$ are also equations.

The quantities on the left of the $=$ constitute the **first member**, and those on the right of the sign $=$ the **second member.**

Name the first member in each equation above; the second member.

The first letters of the alphabet, as a, b, c, etc., are used to represent **known quantities**, and the last letters, as x, y, z, to represent **unknown quantities.**

If the known quantities of an equation are represented by figures, the equation is called a **numerical equation.**

If some or all of the known quantities of an equation are represented by letters, it is called a **literal equation.**

Thus, $5x-8=3x+4$ is a numerical equation, and $3x+a=b-7$ is a literal equation.

Since an equation is an expression of equality between quantities, it is evident that:

1. *The equation will not be affected when any operation is performed on any term by itself which does not change the value of that term.*

Thus in the equation $3x + a = b - 7$, if we write $\frac{14}{2}$ in the place of 7, the equation cannot be affected.

2. *The equation is not destroyed when both members are affected alike.*

Thus in the equation $3x = 6$ if we add the same quantity to each member, the value must be the same, $3x + 2 = 6 + 2$.

Statements like 1 and 2 are called **axioms**, as they require no proof, being self-evident.

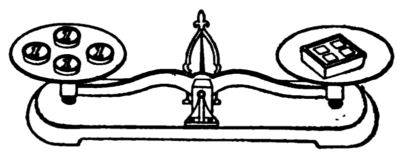
We can illustrate them, however.

The equation has been likened to a pair of scales.

One side of the scales must equal the other in weight if they are to balance. We may say:

Weight of box + its contents = 4 pounds.

It is evident that a two-pound weight may be used in place of two one-pound weights without affecting the balance, or two eight-ounce weights in place of a pound weight, or two of the packages in the box may be tied together, or a package divided, etc., without altering the balance of the scale pans. This illustrates the first axiom.



Again, we may add a pound weight to each side, or take away a pound from each side, or add a pound weight on one side and a pound package on the other, and still the scales remain undisturbed. This illustrates the second axiom.

On these two axioms hang all the laws of the equations.

Take the equation, $\frac{6x}{2} - 5 - 3 + 4x = 9x + 7 - 3 - 5x$.

By the first axiom, we may reduce $\frac{6x}{2}$ to its equal, $3x$; we may add the $3x$ and $4x$ and use the sum, $7x$; we may combine $-5-3$ into -8. We may combine the second member into $4x + 4$. The equation,

now reduced by the first axiom, must still be in perfect balance. We have, $7x - 8 = 4x + 4$.

By the second axiom, we may subtract $4x$ from each member. We then have $7x - 4x - 8 = 4$. By the same axiom, we may add $+8$ to each member, and then have $7x - 4x - 8 + 8 = 4 + 8$.

By the first axiom, we have $3x = 12$.

By the second axiom, dividing each member by 3, we have $x = 4$.

Thus from the first equation we have found the simplest form possible, which gives the value of the unknown quantity.

The changing of terms from one member to the other is called **transposition.**

The uniting of several terms into one or more is called **combination.**

The process of finding the value of the unknown quantity is called **solving the equation.**

[61] Solve the following equation:

	(1)	$3x + 4 + 3 = 9x - 5 - 4x.$
Combining,	(2)	$3x + 7 = 5x - 5.$
Transposing,	(3)	$3x - 5x = -5 - 7.$
Combining,	(4)	$-2x = -12.$
Multiplying by -1,	(5)	$2x = 12.$
Dividing by 2,	(6)	$x = 6.$

If 6 is the right value of x, we can by the first axiom substitute this value of x in the equation without destroying the equality.

The equation is $3x + 4 + 3 = 9x - 5 - 4x$.

Substituting 6 for x, we have

$$3 \times 6 + 4 + 3 = 9 \times 6 - 5 - 4 \times 6.$$
$$18 + 7 = 54 - 5 - 24.$$
$$25 = 54 - 29.$$
$$25 = 25.$$

This is what is called an **identical equation**, because both sides are identical.

The process of proving the value of the unknown quantity is called **verification**.

In each of the following equations see by verification whether the value given to the unknown quantity is correct. If incorrect, find the correct value.

1. $6x-5+43=x-27+100.$
   $x=7.$
2. $9x+8=2x+85.$
   $x=11.$
3. $6x+16=9x-5.$
   $x=7.$
4. $4(x+1)=3(x+2).$
   $x=3.$

Solve the following numeral equations by analyzing each step as in the above solution. Study each step carefully and see that neither axiom is violated. After having found the value of the unknown quantity, verify it as in the model.

1. $7-3x=3-x.$
2. $3x+17=5x+7.$
3. $4x-4+2x=12-3x+x.$
4. $3x-2x+7-3x=3x-4+20-8x.$

[62] 5. $4x-12+\frac{9x}{3}=2x+4+3x.$

6. $8x+24-16+3x=12x+32-4x.$
7. $21x-34-7x-16=3x+10+5x.$
8. $2x+17+5x=21-3x+8-2x.$
9. $x+3x+7+9+5=9x-2x+3x-21.$

If the process of solving equations is now understood, solve the following by the shorter method, generally used; as,

$$4x + 7 + 6x = 3x + 17 + 2x.$$

$$10x + 7 = 5x + 17.$$
$$5x = 10.$$
$$x = 2.$$

VERIFICATION

$$8 + 7 + 12 = 6 + 17 + 4.$$
$$27 = 27.$$

Again, $$\frac{5x}{2} + 4 - x = \frac{6x}{5} + 7.$$

Since multiplying a fraction by its denominator, or any multiple of its denominator, will give an entire quantity (why?), if we multiply each member of the equation by the L. C. M. of the denominators, we shall remove the denominators and not affect the value of the equation. (Why?)

Multiplying each member of the equation by 10 (the L. C. M. of 2 and 5), we have

$$25x + 40 - 10x = 12x + 70.$$

Combining, $$15x + 40 = 12x + 70.$$

Transposing, $$15x - 12x = 70 - 40.$$

Combining, $$3x = 30.$$
$$x = 10.$$

VERIFICATION

$$\frac{5 \times 10}{2} + 4 - 10 = \frac{6 \times 10}{5} + 7.$$
$$25 + 4 - 10 = 12 + 7.$$
$$19 = 19.$$

[63] RULE. *To solve a simple equation, clear the equation of fractions, if any. Transpose all the terms containing the unknown quantity to the first member and the terms containing the known quantities to the second member. Combine and divide by the coefficient of the unknown quantity. Verify.*

Find the value of the unknown quantity in each of the following equations:

1. $x + 6 = 12.$
2. $x - 4 = 2.$
3. $2x + 8 = 14.$
4. $4x - 6 = 22.$
5. $4x - 8 = 2x + 14.$
6. $7x + 15 = 2x + 20.$
7. $8x - 7 = 5x + 14.$
8. $5x - 16 = -x + 8.$
9. $12x - 42 = 5x + 42.$
10. $4x + 4 = 2x + 12.$

[64]

11. $5y + 25 - 2y = 34.$
12. $9x + 11 = 3x + 35.$
13. $10x + 2 - 6x = 50.$
14. $6x - 17 = 4x + 3.$
15. $3x + 7x = 16 + 4.$
16. $2x + 4x - 3x = 27 + 4 - 7.$
17. $3x - 4 = 2x + 6.$
18. $3x + 6 = x + 12.$
19. $8x - 6 = 42.$
20. $3x - 5 = 2x + 30.$
21. $3x + 2 - 7x = 14 - 8x.$
22. $8y + 12 = 3y + 42.$
23. $2y - 9 = 27 - 4y.$
24. $10 + 4x = 7x - 14.$

[65]

25. $9 - 2x = 1 + 6x.$
26. $3 + 7x + 7 = 12x - 15.$
27. $y + 17 = 9y - 7.$
28. $24 - 6y = 33 - 9y.$
29. $124 - 7x = 3x + 64.$
30. $4x + 62 = 9x + 12.$
31. $2x + 74 = 9x - 38.$
32. $12x - 62 = 8x - 22.$
33. $7x - 33 = x + 9.$
34. $3x + 40 = 8x - 5.$
35. $17x - 37 = 14x + 2.$
36. $21x - 30 = 11x - 10.$
37. $6x - 7 = 10x + 9.$
38. $34 - 6x = 3x - 11.$
39. $140 + 4x = 9x + 50.$
40. $84 - 12x = 30 + 6x.$

[66]

41. $6x + 120 = 12x - 120.$
42. $32 + 7x = 11x - 12.$
43. $x - 17 = 19 - 2x.$
44. $5(x - 2) = 3(4x - 8).$
45. $3(x+2) = 5(2x-12) - 4.$
46. $7(3x-4) = 3(6x-9) + 2.$
47. $7(2x + 3) = 3(6x + 4) - 7.$
48. $4(x + 7) = 6(x - 2) + 32.$
49. $8(6x + 3) = 3(12x - 10) - 6.$
50. $2(7x + 17) = 9(4x - 20) + 16.$
51. $2(x + 1) + 4(x + 2) = 5(3x - 2) - 16.$
52. $2(2x - 2) + 4(4x - 4) = 6(6x - 6).$
53. $9(7x - 3) + 9 = 6(4x + 4) + 3(6x + 7).$

FRACTIONAL EQUATIONS

[67] In the following equations, find the value of the unknown quantity, by transposing and combining, without clearing of fractions:

54. $\frac{1}{3}x + 6 = 12 - \frac{2}{3}x.$
55. $\frac{4}{5}x - 8 = 7 - \frac{1}{5}x.$
56. $\frac{3}{8}x + 3 = 11 - \frac{1}{8}x.$
57. $\frac{1}{3}x + \frac{1}{4}x = \frac{3}{4}x - 2.$
58. $\frac{3}{4}x - \frac{3}{8}x = \frac{1}{4}x + 3.$
59. $26 - \frac{3}{5}x = \frac{7}{10}x.$
60. $9 - \frac{x}{5} + \frac{3x}{10} = \frac{4x}{5} + 2.$
61. $\frac{1}{2}x - 6 - \frac{1}{3}x + 3 = \frac{2}{3}x - \frac{3}{4}x + 6 - \frac{1}{2}x.$
62. $\frac{3}{4}x + 17 - \frac{1}{2}x = 12 + 1\frac{1}{2}x - \frac{1}{4}x.$
63. $2.1x - 16 = 39 - 3.4x.$

[68] Solve the following by first clearing of fractions (see page 76):

64. $$\frac{x}{2}-6-\frac{x}{3}+3=\frac{2x}{3}-\frac{3x}{4}+6-\frac{x}{2}.$$

This is the same as Example **61**, page 78.

65. $$\frac{3x}{4}+17-\frac{x}{2}=12+\frac{3x}{2}-\frac{x}{4}.$$

This is the same as Example **62**, page 78.

66. $$\frac{21x}{10}-16=39-\frac{34x}{10}.$$

This is the same as Example **63**, page 78. Which method is the easier?

OPTIONAL WORK

67. $$\frac{5x}{6}-\frac{3x}{8}=\frac{7x}{24}+2.$$

68. $$\frac{x}{4}+\frac{x}{5}-\frac{x}{6}=\frac{x}{10}+\frac{11}{2}.$$

69. $$\frac{3x}{2}+\frac{1}{2}=\frac{2x}{3}+\frac{3x}{4}.$$

70. $$\frac{5x}{6}+\frac{2x}{3}-65=\frac{3x}{8}+\frac{7x}{12}.$$

71. $$x+\frac{x}{4}+\frac{3x}{7}-10=\frac{4x}{7}+\frac{3x}{4}.$$

72. $$\frac{x+4}{3}=3+\frac{x-1}{7}.$$

73. $$\frac{6x-4}{9}-\frac{3x+2}{18}=\frac{4}{9}.$$

(Be careful about the – sign before the fraction.)

74. $\frac{x-6}{4}-\frac{x-4}{6}=5-\frac{x}{8}.$

75. $\frac{3x+3}{5}-\frac{x-2}{10}=\frac{9x}{20}+\frac{9}{10}.$

76. $\frac{2x-3}{4}-\frac{3x-5}{6}=\frac{5x-1}{12}-\frac{18}{12}.$

77. $\frac{x}{3}+\frac{x-7}{4}=7.$

78. $\frac{x}{4}+\frac{3x-2}{8}=41.$

79. $\frac{4x-6}{3}-\frac{5x}{6}=2.$

80. $2x-7+3x=\frac{20+5x}{10}.$

PROBLEMS

In the solution of problems there are two distinct steps.

1st. The statement of the problem; that is, expressing the conditions of the problem in algebraic language and concluding with an equation involving those conditions.

2d. The solution of the equation; that is, finding the value of the unknown quantity.

The first part is not algebra proper, although it is necessary in order to present an equation for solution.

To state a problem properly requires knowledge of the subject and ingenuity in combining the conditions, so as to secure as simple an equation as possible.

You should read the problem over very carefully. Do not hastily start, "Let $x =$" the first unknown quantity you see in the problem. Rather study all the conditions and see whether, having the answer given, you can prove it.

Remember, also, that each member of the equation must represent the same kind of units. x must equal some number; some number of sheep, of dollars, years, miles, etc.

1. A farmer bought 4 lambs and 7 sheep for $54. If a sheep cost twice as much as a lamb, how much did each cost?

MODEL SOLUTION

Let x = number of dollars each lamb cost,

$2x$ = number of dollars each sheep cost.

$$4x + 7(2x) = 54.$$
$$4x + 14x = 54.$$
$$18x = 54.$$

$x = 3$, number of dollars each lamb cost.

$2x = 6$, number of dollars each sheep cost.

VERIFICATION

$$4 \times 3 + 7 \times 6 = 54.$$
$$12 + 42 = 54.$$

2. A boy bought an algebra and a dictionary, paying $4.50 for both. If the dictionary cost 8 times as much as the algebra, how much did each cost?

3. A company of 294 persons consisted of men, women, and children. If there were twice as many women as children, and 4 times as many men as children, how many were there of each?

4. Frank's age is twice his sister's age, and the sum of their ages is 30 years. How old is each?

5. The sum of two numbers is 120, and the larger is 4 times the smaller. What are the numbers?

[69] 6. The sum of two numbers is 75 and their difference is 15. What are the numbers?

SUGGESTION. The larger of two numbers = the smaller + the difference.

7. Find two numbers such that their difference is 12, and if 25 is added to their sum, the amount will be 75.

8. Find two numbers whose sum is 150 and whose difference is equal to the smaller.

9. Find two numbers such that their sum is 160 and the larger equals 3 times the smaller.

10. Find two numbers whose difference is 80 and one of which is 3 times as large as the other.

11. Find two numbers whose sum is 4 times the smaller, and whose difference is 40.

12. Find two numbers whose sum is 150 and whose difference is equal to $\frac{1}{2}$ the larger.

[70] 13. A man's horse and carriage are worth $360. The horse is worth $60 more than the carriage. How much is each worth?

14. A merchant failing in business owes A twice as much as he owes B, and he owes C as much as he owes A and B. If he owes $5400 in all, how much does he owe each?

Will it be best to let x = what he owes to A?

15. The ages of four children are just 3 years apart. If the sum of their ages is 90, what are their ages?

16. A man left his property to his wife, son, and daughter, leaving his son $\frac{1}{2}$ as much as his wife, and his daughter $\frac{1}{2}$ as much as his son. How much did he leave each, if his property was worth $70,000?

To avoid fractions, let x = the sum he left his daughter.

17. A man having no children left his property to his wife, his brother, and his sister. To his sister he gave $\frac{1}{2}$ as much as to his wife, and to his brother he gave $\frac{1}{2}$ as much as to his sister. If the wife received $30,000 more than the brother, how much was the property worth?

18. A man spends $\frac{1}{4}$ of his yearly income for board and $\frac{1}{3}$ for other expenses, and saves $500. What is his income?

19. Divide a stick 36 inches long into two parts so that the shorter part shall be just $\frac{4}{5}$ as long as the other.

20. A certain number diminished by its $\frac{1}{3}$ and $\frac{1}{4}$ equals 25. What is the number?

[71] **21.** A man, being asked his age, replied that if his age were diminished by its $\frac{1}{4}$ and 10 years it would be equal to 20 years. What was his age?

22. If Mr. Frank's age is increased by its $\frac{1}{3}$ and $\frac{1}{5}$, the sum will be 92 years. What is his age?

23. From a barrel of oil $\frac{1}{6}$ leaked out; then 8 gallons were sold. The barrel was then $\frac{2}{3}$ full. How many gallons were in it at first?

24. I am thinking of a number. If I double it and then take away $\frac{1}{4}$ of the product, I shall have 24. What number have I in mind?

25. Six times a number diminished by 12 equals 3 times the number. What is the number?

26. Three times a number increased by 100 equals 8 times the number. What is the number?

27. Divide $200 between Charles and Frank so that Charles shall have $40 more than Frank.

28. A prize of $4500 was divided among three racers, A receiving $500 more than B, and B $500 more than C. How much did each receive?

[72] **29.** If a certain number is doubled, and then increased by 20, the result is 14 less than 4 times the number. Find the number.

30. Find a number whose double exceeds its half by 6.

31. Find a number whose half is 12 less than its $\frac{3}{4}$.

32. A man earns $1 a day more than his son. In 10 days they together earn $40. What are the daily wages of each?

33. A boy earns 50 cents a day less than his father, and in 20 days the father has earned $20 more than the son earns in 10 days. Find the daily wages of each.

34. The deposits in a bank during three days amounted to $14,000. If the deposits each day, after the first, were double those of the preceding day, how much was deposited each day?

35. The sum of three consecutive numbers is 63. What are the numbers?

36. I have as many dollars in my right-hand pocket as I have quarters in my left-hand pocket. If I transfer six dollars from my right-hand pocket to my left-hand pocket, I shall have the same amount of money in each pocket. How many dollars and how many quarters have I?

PERCENTAGE

[73] 1. A man spends $33\frac{1}{3}\%$ of his salary for board, 25% for clothes, and 20% for incidentals. If he saves \$260, what is his income?

(Transform percentage values into equivalent common fractions.)

2. A man spends 40% of his income for board, $33\frac{1}{3}\%$ of the remainder for clothes, and \$250 for incidentals. If he saves \$550, how much is his income?

3. A man sold a horse for \$300, thereby clearing 25% of the cost. How much did it cost?

4. A man sold a house for \$4000, and lost 20% by the transaction. How much did the house cost?

5. A merchant gained $16\frac{2}{3}\%$ by selling a pair of skates for \$5.60. How much did they cost?

6. A farmer sold 10 cows at \$63 each, at a gain of 40%. How much did he pay for his cows, and how much did he gain?

7. A man bought a farm and sold it so as to clear \$200. If he made 10% by the transaction, how much did he get for the farm?

8. A merchant sold goods at a profit of $16\frac{2}{3}\%$. If he cleared \$16 on a set of furniture, how much did he get for it?

9. If a man bought a house for \$2400 and sold it for \$3000, what per cent did he gain?

$\left(\text{Let } \frac{x}{100} = \text{rate.}\right)$

10. A man bought a horse for \$250 and sold it for \$300. What per cent did he gain?

INTEREST

[74] You have learned in interest that the principal multiplied by the rate multiplied by the time (in years) will give the interest. Or,

$$P \times R \times T = I.$$

Having any three of these quantities given, the fourth may easily be found by solving this equation for the unknown quantity.

I. To find the interest.

What is the interest of \$250 for 5 years at 4%?

Let $x = \text{interest}.$

General equation, $PRT = I.$

Substituting, $250 \times \frac{4}{100} \times 5 = x.$

$$5000 = 100\,x.$$

$$x = 50.$$

II. To find the principal.

What principal will give \$90 interest in 3 years at 6%?

Let $x = \text{principal}.$

General equation, $PRT = I.$

Substituting, $x \times \frac{6}{100} \times 3 = 90.$

$$18\,x = 9000.$$

$$x = 500.$$

III. To find the rate.

At what rate will $700 give $210 interest in 7 years 6 months?

Let $x = \text{rate}.$

General equation, $PRT = I.$

Substituting, $700 \times \frac{x}{100} \times \frac{15}{2} = 210.$

$$10500\,x = 42000.$$

$$x = 4.$$

IV. To find the time.

In what time will $400 give $240 interest at 5%?

Let $x = \text{time in years}.$

General equation, $PRT = I.$

Substituting, $400 \times \frac{5}{100} \times x = 240.$

$$2000\,x = 24000.$$

$$x = 12.$$

The above are the four cases of interest. Apply the formulas to the following examples:

1. Find the interest of $175 at 4% for 7 years.

2. Find the interest of $324 at 4% for 3 years.

[75] 3. Find the interest of $1250 for 6 years at $3\frac{1}{2}$%.

4. Find the interest of $650 for 4 years 6 months at 4%.

5. Find the interest of $480 for 3 years 6 months at 5%.

6. What principal will in 3 years at 5% give $300 interest?

7. What principal will in 10 years at 6% give $300 interest?

8. At what rate will $300 give $60 interest in 4 years?

9. In what time will $500 give $40 interest at 4%?

10. In what time will $800 give $160 interest at 5%?

OPTIONAL WORK

1. Find the interest of $720 for 4 yr. 6 mo. at 5%.

2. Find the interest of $850 for 3 yr. 3 mo. at 4%.

3. Find the interest of $1200 for 5 yr. 8 mo. at 6%.

4. What principal will in 3 years at 5% give $60 interest?

5. The interest on a certain sum of money for 7 years 6 months at 6% is $360. Find the principal.

6. What principal will give $180 interest in 4 years at $4\frac{1}{2}$%?

7. At what rate will $200 give $114 interest in 9 years 6 months?

8. At what rate will $600 amount to $708 in 4 years?

9. How long will it take for $300 to gain $60 interest at 4%?

10. If the principal is $1000, interest $270, and rate 6%, find the time.

ORIGINAL WORK

Select problems under parallel cases in percentage and interest in arithmetic and solve by algebra.

SECOND HALF YEAR

ADDITION AND SUBTRACTION

[1–7] Review pages 24 to 37, taking the "Optional Work." (Allow seven lessons for this review.)

EXAMPLES FOR ADVANCED WORK IN ADDITION AND SUBTRACTION

[8] **1.** Add $9 + x$ and $6 - y$.

2. Add $2a - 2 + 3a + 4b - 1$, $2a - 3b + 2b - 1$, $4b - 2a - 2 - 5b - 1$, $8a - 3a - 2 + 2b$, and $6a^2 + 3b - 1 + 4a$.

3. Add $5x^2y - xy^2 - 3x^3$, $3xy^2 - 2y^3 + x^3$, $8x^3 + 5y^3$, and $9x^2y - 2x^3 + xy^2$.

4. Add $my^2 - 9ny + 7p$, $-2my^2 + 3ny - 6p$, $7ny - 4p$, and $3my^2$.

5. Add $7a^{\frac{1}{2}}y - 2ay^{\frac{1}{2}} + 7$, $a^{\frac{1}{2}}y + 3ay^{\frac{1}{2}}$, and $2 + 3a^{\frac{1}{2}}y - ay^{\frac{1}{2}}$.

6. Add $3(a - c)(x + y)$, $a(x + y)$, $5c(x + y)$, and $2(a - c)(x + y)$. (What is the common term?)

7. Add $\frac{1}{3}ab - \frac{1}{2}ay + 2xy - \frac{2}{3}ax$, $\frac{2}{3}ay - \frac{3}{4}xy$, $\frac{1}{2}ay - \frac{3}{2}xy + \frac{2}{3}ab$, and $\frac{1}{2}ax + 2ay - \frac{1}{4}ab$.

8. Subtract the sum of $\frac{1}{2}a + b - \frac{1}{3}c$, $a - \frac{2}{3}b + \frac{1}{2}c$, and $\frac{1}{3}b - \frac{3}{2}a - \frac{1}{4}c$ from $a + b - c$.

MULTIPLICATION

[9] Review multiplication, pages 39 to 45.

To multiply when the multiplier is a binomial.

Multiply 2467 by 42.

This means to multiply $2000 + 400 + 60 + 7$ by $40 + 2$.

We multiply each term of the multiplicand by the 2 and then by the 40 (or four tens). In practice, however, we multiply the figures in the different orders by each term of the multiplier, keeping the orders of the partial products in their proper columns.

In algebra the operation is much simpler, as there is no carrying.

Multiply $x^2 + y^2 - xy$ by $y + x$.

For convenience the multiplicand and multiplier should be arranged in the same order and according to the exponents of the letter selected. We can arrange this example according to x or y, but algebraic expressions are generally arranged in the order of letters of the alphabet.

The multiplicand arranged according to x would be $x^2 - xy + y^2$. The multiplier would be $x + y$.

$$
\begin{array}{l}
x^2 - xy + y^2 \\
\underline{x + y \qquad\qquad} \\
x^3 - x^2y + xy^2 \\
\underline{\qquad\ \ x^2y - xy^2 + y^3} \\
x^3 \qquad\qquad\qquad + y^3
\end{array}
$$

In algebra the multiplier is placed at the left for convenience, as we generally start to multiply by the left-hand term.

In arithmetic each term of the multiplicand is multiplied by each term of the multiplier. For the same reason, in algebra, each term of the multiplicand must be multiplied by each term of the multiplier. The partial products are arranged in columns containing similar terms, for convenience in adding. In this particular example some of the partial products cancel each other, and the product is $x^3 + y^3$. We readily see that this case is but an extension of the case of multiplying by monomials.

Rule. ***To multiply a polynomial by a polynomial, multiply each term of the multiplicand by each term of the multiplier separately, and add the partial products.***

Multiply:

1. $a + 4$ by $a + 10$.
2. $x - 11$ by $x - 2$.
3. $x + y + 1$ by $x - 1$.
4. $-x + 3$ by $-x - 3$.
5. $a^2 - ab + b^2$ by $a - b$.

[10] 6. $2a^3 + 4a^2 + 8a + 16$ by $3a - 6$.

7. $3x + y$ by $x + 2y$.
8. $2a^3 + 8a - 4$ by $a^2 - 4$.
9. $2x^3 + 3x^2 + 4x + 1$ by $3x - 5$.
10. $a^2 - 50a + 100$ by $a + 2$.
11. $4x^2y + 2xy^2 + 1$ by $x + y$.
12. $3ax + 3a + x + 1$ by $2a - 1$.
13. $2b^2 - 2b - 2$ by $b + 2$.
14. $x + xy - y^2$ by $x - y$.
15. $m - m^2 + n - n^2$ by $m - n$.
16. $x^2 - x + 1$ by $x + 1$.
17. $3ax + 4a + 4$ by $ax - 1$.

[11] 18. $x + y - a$ by $x + a$.

19. $3x + 4$ by $x - 4$.
20. $2ac + 3c + 4$ by $ac - 2$.

ORIGINAL WORK

Make up five examples in multiplication and work out.

OPTIONAL WORK

Multiply:

1. $1 + x + x^2 + x^3 + x^4$ by $1 - x$.
2. $4a^2 + 6ab + 9b^2$ by $2a - 3b$.
3. $16x^2 - 4xy - 20y^2$ by $3x - 5y$.
4. $2x^3 + 2x^2y + 2xy^2 + 2y^3$ by $x - y$.
5. $x^n + y^n$ by $x^n - y^n$.
6. $x^n + y^n$ by $x - y$.
7. $-x^2 + 9x + 10$ by $-x - 4$.
8. $x^{\frac{2}{3}} - x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}}$ by $x^{\frac{1}{3}} + y^{\frac{1}{3}}$.

SPECIAL CASES IN MULTIPLICATION

[12] Preserve the answers.

Set I

Multiply.

1. $x + y$ by $x + y$.
2. $a + b$ by $a + b$.
3. $c + d$ by $c + d$.
4. $m + n$ by $m + n$.
5. $2x + 3$ by $2x + 3$.

Compare these five answers as to number of terms, signs, and exponents.

Set II

1. $x - y$ by $x - y$.
2. $a - b$ by $a - b$.
3. $c - d$ by $c - d$.
4. $m - n$ by $m - n$.
5. $2x - 3$ by $2x - 3$.

How do these examples differ from those in Set I? Compare these answers as you did those in the previous set. How do they compare with those in Set I?

Set III

1. $x + y$ by $x - y$.
2. $a + b$ by $a - b$.
3. $c + d$ by $c - d$.
4. $m - n$ by $m + n$.
5. $2x + 3$ by $2x - 3$.

How do these examples compare with the previous five? How do the answers compare with one another?

[13] **To find the square of the sum of two quantities.**

By studying examples in Set I you see that you are to multiply a binomial consisting of the sum of two quantities by itself, or square the binomial.

Observe:

I. That the result is always a trinomial.

II. That the first and third terms of the product are the squares of the first and second terms of the binomial.

III. That the second term of the product is twice the product of the terms of the binomial and is positive. Hence we have the following principle:

PRINCIPLE. *The square of the sum of two quantities is equal to the square of the first plus twice the product of the first by the second plus the square of the second.*

FORMULA: $(a + b)^2 = a^2 + 2ab + b^2$.

Write out the answers to the following examples without multiplying. If you cannot do this, work again the examples in Set I, studying the process and the results carefully with reference to the principle.

1. $(x+a)(x+a)$.
2. $(m+n)(m+n)$.
3. $(a+1)(a+1)$.
4. $(a+7)(a+7)$.
5. $(x+2)^2$.
6. $(a+x)^2$.
7. $(6+4)^2$.
8. $(3x+y)^2$.
9. $(a^2+y^2)^2$.
10. $(2x+3y)^2$.
11. $(3a+4)^2$.
12. $(2x+4)(2x+4)$.
13. $(3a+5)(3a+5)$.
14. $(2x+\frac{1}{2})^2$.
15. $(4x+\frac{1}{4}y)^2$.
16. $(d+3c)^2$.

ORIGINAL WORK

Make up three examples like the above and write the products or squares.

[14] To find the square of the difference of two quantities.

By studying the examples in Set II you notice that you are to find the square of the difference of two quantities, instead of the square of the sum of two quantities, as in the first set. How do the products of the two sets compare? Deduce the following principle:

PRINCIPLE. *The square of the difference of two quantities is equal to the square of the first minus twice the product of the first by the second plus the square of the second.*

FORMULA: $(a-b)^2 = a^2 - 2ab + b^2$.

Write out the following results without multiplying. If you cannot do this readily, work the examples in Set II again until you can apply the principle and formula readily.

1. $(x-y)(x-y)$.
2. $(x-a)(x-a)$.
3. $(a-5)^2$.
4. $(x-7)^2$.
5. $(7-2)^2$.
6. $(a-3x)^2$.
7. $(2a-c)^2$.
8. $(2a-4)^2$.
9. $(3a-2x)^2$.
10. $(ab-4)^2$.
11. $(4x-2y)^2$.
12. $(2a-3b)^2$.
13. $(a^2-b^2)^2$.
14. $(a-\frac{1}{2}b)^2$.
15. $(2x-\frac{1}{4}y)^2$.
16. $(\frac{1}{2}-2y)^2$.

ORIGINAL WORK

Make up three examples like the above and write the products or squares.

[15] To find the product of the sum and the difference of two quantities.

Study the examples in Set III and see whether you can deduce the following principle:

PRINCIPLE. *The product of the sum and the difference of two quantities is equal to the difference of their squares.*

FORMULA: $(a+b)(a-b)=a^2-b^2$.

Write out the following products without multiplying:

1. $(a+b)(a-b)$.
2. $(m+n)(m-n)$.
3. $(x+y)(x-y)$
4. $(a+3)(a-3)$.
5. $(2a+4)(2a-4)$.
6. $(4x+y)(4x-y)$.

7. $(2a+6)(2a-6)$.
8. $(a+2y)(a-2y)$.
9. $(9+3)(9-3)$.
10. $(m^2+n^2)(m^2-n^2)$.
11. $(a+3x)(a-3x)$.
12. $(x+\frac{1}{2}y)(x-\frac{1}{2}y)$.
13. $(x^2-x)(x^2+x)$.
14. $(2a^2-3y^2)(2a^2+3y^2)$.

ORIGINAL WORK

Make up three examples like the above and write the products.

[16] Product of two binomials having a common term.

$$\begin{array}{r} x+2 \\ \underline{x+6} \\ x^2+2x \qquad \\ \underline{\quad 6x+12} \\ x^2+8x+12 \end{array} \qquad \begin{array}{r} x+2 \\ \underline{x-6} \\ x^2+2x \qquad \\ \underline{\quad -6x-12} \\ x^2-4x-12 \end{array} \qquad \begin{array}{r} x-2 \\ \underline{x-6} \\ x^2-2x \qquad \\ \underline{\quad -6x+12} \\ x^2-8x+12 \end{array} \qquad \begin{array}{r} x-2 \\ \underline{x+6} \\ x^2-2x \qquad \\ \underline{\quad 6x-12} \\ x^2+4x-12 \end{array}$$

From these examples we may deduce the following principle:

PRINCIPLE. *In the product of two binomials having a common term the first term of the product is the square of the common term; the second term is the algebraic sum of the unlike terms multiplied by the common term; the third term is the product of the unlike terms.*

Write the product of the following quantities:

1. $(x+3)(x+5)$.
2. $(x+5)(x+7)$.
3. $(a+6)(a+4)$.
4. $(x+4)(x-1)$.
5. $(a+7)(a-3)$.
6. $(b+9)(b-3)$.
7. $(m+8)(m-2)$.
8. $(y+2)(y-4)$.
9. $(x+1)(x-6)$.
10. $(c+4)(c-9)$.

[17]

11. $(a+3)(a-4)$.
12. $(a-2)(a-3)$.
13. $(x-4)(x-7)$.
14. $(y-3)(y-6)$.
15. $(x-2)(x-9)$.
16. $(3-8)(3+20)$.
17. $(1-4b)(1-2b)$.
18. $(a+5)(a-3)$.
19. $(b+9)(b+2)$.
20. $(b-10)(b+3)$.

[18]

21. $(a+7)(a+4)$.
22. $(x+12)(x-2)$.
23. $(a-4)(a-12)$.
24. $(c-2)(c-5)$.
25. $(x-a)(x-2a)$.
26. $(x-2a)(x+4a)$.
27. $(x+a)(x-3a)$.
28. $(x+2a)(x+3a)$.
29. $(2x+4)(2x+2)$.
30. $(3a+7)(3a-6)$.
31. $(4+x)(4+y)$.
32. $(6-a)(6-b)$.
33. $(6x+12)(6x-2)$.
34. $(x^2-3)(x^2-2)$.

OPTIONAL WORK

Make up three examples like the above and write the products.

DIVISION

[19–20] Review Division by Monomials, pages 46 to 51. Take optional problems (one lesson).

When the divisor is a binomial.

Divide 4615 by 71.

```
71)4615(65
   426
    355
    355
```

We find the first figure, or term, in the quotient by "trying" the first term of the divisor into the first term or terms of the dividend. 7 into 46 will give a quotient of 6, the first term of the quotient. Multiplying the divisor by the first figure of the quotient, subtracting and bringing down another term of the dividend, we have 355 for a new dividend.

Again we "try" the first term of the divisor into the first term or terms of the dividend, and find that 7 will go into 35, 5 times. Multiplying the divisor by the new term of the quotient, 5, and subtracting, we find the division is complete and exact.

Now in algebra the process is almost exactly the same.

Divide $a^3 - 5a + 12 + 2a^2$ by $4 + a$.

We must first arrange the expressions in order of the same letter.

In algebra, for convenience in multiplying and to save space, we write the divisor at the right of the dividend and the quotient below, as follows:

$$\begin{array}{rrrr|l}
a^3 & +2a^2 & -5a & +12 & a+4 \\
\underline{a^3} & \underline{+4a^2} & & & \overline{a^2-2a+3} \\
 & -2a^2 & -5a & & \\
 & \underline{-2a^2} & \underline{-8a} & & \\
 & & 3a & +12 & \\
 & & \underline{3a} & \underline{+12} &
\end{array}$$

Dividing the first term of the dividend by the first term of the divisor (a), we have a^2, the first term of the quotient.

We multiply the divisor by this term, a^2, and subtract the product from the dividend. The remainder is $-2a^2$. We bring down another term of the dividend for a new dividend. Dividing the first term of the new dividend by the first term of the divisor (a), for the

second term of the quotient we get $-2a$. Multiplying the whole divisor by $-2a$, we have $-2a^2-8a$. Subtracting, we have a remainder of $3a$. Bringing down the remaining term of the dividend, we have $3a+12$ for a new dividend. Dividing the first term by the first term of the divisor (a), we have 3 for the third term of the quotient. Multiplying the whole divisor by 3 and subtracting, there is no remainder. The division is, therefore, exact, and a^2-2a+3 is the quotient. Prove it.

The process is the same, no matter how many terms there are in the divisor.

Divide a^3-b^3 by $a-b$.

$$
\begin{array}{l|l}
a^3-b^3 & a-b \\
\underline{a^3-a^2b} & \overline{a^2+ab+b^2} \\
\quad a^2b-b^3 & \\
\quad \underline{a^2b-ab^2} & \\
\qquad ab^2-b^3 & \\
\qquad \underline{ab^2-b^3} &
\end{array}
$$

In this example the second term in each product is unlike any term in the dividend. It is brought down first because the expression is arranged according to a, and the term containing a should precede the term having no a in it. This course follows until the division is completed. If there is a remainder, it is added to the quotient as unperformed division, the same as in arithmetic. For example,

$$
\begin{array}{l|l}
x^3+3x^2+3x-3 & x+2 \\
\underline{x^3+2x^2} & \overline{x^2+x+1-\dfrac{5}{x+2}} \\
\quad x^2+3x & \\
\quad \underline{x^2+2x} & \\
\qquad x-3 & \\
\qquad \underline{x+2} & \\
\qquad -5 &
\end{array}
$$

RULE. *To divide a polynomial by a polynomial, arrange the dividend and divisor according to the powers of the same letter. Divide the first term of the dividend by the first term of the divisor for the first term of the quotient. Multiply the*

divisor by this term of the quotient, and subtract the product from the dividend. To the remainder annex one or more terms of the dividend for a new dividend. Divide as before, and continue the process as far as possible.

If there is a remainder, express the uncompleted division in the form of a fraction and annex it to the quotient with the proper sign.

Perform the multiplication indicated in Example 6 on page 92. Then divide the product by the multiplier and see whether the quotient equals the multiplicand.

[21] Divide:

1. $a^2 + 2ab + b^2$ by $a + b$.
2. $x^2 - 2xy + y^2$ by $x - y$.
3. $x^2 - x - 12$ by $x - 4$.
4. $x^2 - 11x + 28$ by $x - 7$.
5. $a^3 - 3a^2b + 3ab^2 - b^3$ by $a - b$.
6. $a^3 + b^3$ by $a + b$.
7. $2x^3 - 19x^2 + 26x - 16$ by $x - 8$.
8. $a^5 - b^5$ by $a - b$.
9. $x^2 + 2x - 3$ by $x + 3$.
10. $x^2 - 11x + 24$ by $x - 8$.

[22] 11. $a^2 + 11a + 28$ by $a + 4$.

12. $b^2 - 4b - 12$ by $b - 6$.
13. $m^2 - 9m - 22$ by $m - 11$.
14. $c^4 - 8c^2 + 15$ by $c^2 - 3$.
15. $d^2x^2 + 6dxy - 9dx - 54y$ by $dx + 6y$.

16. $a^4 + 8a^2 - 9$ by $a^2 - 1$.

17. $a^4c^4 - 10a^2c^2 + 24$ by $a^2c^2 - 6$.

18. $x^6 + 9x^3 - 36$ by $x^3 + 12$.

19. $x^4 - y^4$ by $x^2 - y^2$.

[23] 20. $x^4 - y^4$ by $x - y$.

21. $a^4 + b^4$ by $a + b$.

22. $16x^4 - 81$ by $2x + 3$.

23. $16x^4 - 81$ by $2x - 3$.

24. $12a^2 + 13a - 14$ by $3a - 2$.

25. $x^2 - 2xy + y^2 - xz + yz$ by $x - y$.

26. $xy + yz + ax + az$ by $x + z$.

27. $6a^3 - 16a^2b + 6ab^2 + 4b^3$ by $2a - 4b$.

OPTIONAL WORK

Divide:

1. $a^2 + 7ab + 10b^2$ by $a + 2b$.

2. $20c^2 - 32cd + 12d^2$ by $5c - 3d$.

3. $x^8 - 2x^4y^4 + y^8$ by $x^4 - y^4$.

4. $a^2 - 4a + a^3 + 80$ by $a + 5$.

5. $a^3 - 2ab^2 + b^3$ by $a - b$.

6. $3(m+n)^2 + 4(m+n)$ by $(m+n)$.

7. $x - y$ by $x^{\frac{1}{2}} + y^{\frac{1}{2}}$.

8. $a^3 + 1$ by $a + 1$.

9. $x^{\frac{2}{3}} - y^{\frac{2}{3}}$ by $x^{\frac{1}{3}} - y^{\frac{1}{3}}$.

10. $a^{2m} - b^{2m}$ by $a^m - b^m$.

11. $a^4 - a^3 - a - 1$ by $a^2 + 1$.

SPECIAL CASES IN DIVISION

[24] **Set I.** Divide:

1. a^2+b^2 by $a+b$.
2. a^4+b^4 by $a+b$.
3. a^2+b^2 by $a-b$.
4. a^4+b^4 by $a-b$.

Notice in these four examples that the divisor is not exactly contained in the dividend.

PRINCIPLE. *The sum of the same even powers of two quantities is not divisible by the sum or by the difference of the quantities.*

ORIGINAL WORK

Make up an example like the above and divide.

Set II. Divide:

1. a^2-b^2 by $a+b$.
2. a^4-b^4 by $a+b$.
3. a^2-b^2 by $a-b$.
4. a^4-b^4 by $a-b$.

Preserve these answers and those in Sets III and IV, written out in neat form.

PRINCIPLE. *The difference of the same even powers of two quantities is divisible by the sum and by the difference of the quantities.*

Set III. Divide:

1. a^3+b^3 by $a+b$.
2. a^5+b^5 by $a+b$.
3. a^3+b^3 by $a-b$.
4. a^5+b^5 by $a-b$.

PRINCIPLE. *The sum of the same odd powers of two quantities is divisible by the sum of the quantities but not by their difference.*

ORIGINAL WORK

[25] Make up one example like those in Set III and work it out.

Set IV. Divide:

1. $a^3 - b^3$ by $a - b$.
2. $a^5 - b^5$ by $a - b$.
3. $a^3 - b^3$ by $a + b$.
4. $a^5 - b^5$ by $a + b$.

PRINCIPLE. *The difference of the same odd powers of two quantities is divisible by the difference of the quantities, but not by their sum.*

ORIGINAL WORK

Make up two examples like those in Set IV.

These principles may be combined as follows:

1. **Even Powers.** — *The sum of the same even powers of two quantities is divisible neither by the sum nor by the difference of the quantities; but the difference of the same even powers of two quantities is divisible both by the sum and by the difference of the quantities.*

2. **Odd Powers.** — *The sum of the same odd powers of two quantities is divisible only by the sum of the quantities; and the difference of the same odd powers of two quantities is divisible only by the difference of the quantities.*

[26] Examine your answers to the examples in Sets II, III, IV.

How many terms are there in the quotient compared with the exponent, or power, of the dividend? How do the letters run in the several quotients? How do the signs run? Can you write out the quotients of the following without dividing? If not, divide by long division, comparing with the proper examples in the previous sets.

1. $x^3 + y^3$ divided by $x + y$.
2. $x^4 - y^4$ divided by $x + y$.
3. $x^4 - y^4$ divided by $x - y$.

ORIGINAL WORK

Make up three examples like those on pages 102 and 103, using m and n for the quantities.

[27] What binomials will divide the following? Write the quotients without dividing. Study carefully, seeing which set each example comes under.

1. $a^3 + c^3$.	4. $a^3 - b^3$.	7. $x^4 - y^4$.	10. $x^4 + y^4$.
2. $x^2 - y^2$.	5. $b^5 + c^5$.	8. $x^3 - y^3$.	11. $m^3 + n^3$.
3. $a^4 + y^4$.	6. $a^5 - y^5$.	9. $x^2 + y^2$.	12. $m^4 - n^4$.

ZERO EXPONENT AND NEGATIVE EXPONENTS

[28] Divide a^2 by a^2.

If we divide by subtracting the exponents, we have

$$a^2 \div a^2 = a^{2-2} = a^0.$$

If we express the division in the form of a fraction and reduce to lowest terms, we have

$$\frac{a^2}{a^2} = \frac{1}{1} = 1.$$

Hence, $$a^0 = 1.$$

We can make this general by using a^n; that is, any quantity with any exponent.

$$a^n + a^n = a^{n-n} = a^0.$$

$$\frac{a^n}{a^n} = 1.$$

Hence, $$a^0 = 1.$$

That is,

Any quantity with zero exponent is equal to 1.

If the exponent of a letter is the same in dividend and divisor, the letter does not appear in the quotient. (Why?)

If the exponent of a letter in the divisor is greater than the exponent of that letter in the dividend, the exponent of that letter in the quotient will be negative.

This must necessarily follow, since the exponent of that letter in the quotient is equal to the exponent of the letter in the dividend minus the exponent of the letter in the divisor.

Divide $9\,a^2bc$ by $3\,a^4b^2c$.

By subtracting the exponents in the process of division, we have

$$3\,a^4b^2c)\,9\,a^2bc\,(3\,a^{2-4}b^{1-2}c^{1-1} \text{ or } 3\,a^{-2}b^{-1}.$$
$$\underline{9\,a^2bc}$$

If a letter appears in the divisor and not in the dividend, it is written in the quotient with the sign of the exponent changed.

Divide $\dfrac{8\,a^3}{2\,a^2b}$.

Since there is no b in the dividend, b^0 (being equal to 1) may be written in.

$$\frac{8\,a^3b^0}{2\,a^2b} = 4\,ab^{0-1} = 4\,ab^{-1}.$$

Divide:

1. $32\,a^3y^2x$ by $16\,a^4y^2x^3$.
2. $26\,x^2y^2$ by $13\,x^3y^2z^3$.
3. $24\,a^{m+2}$ by $3\,a^{2m}$.
4. $24\,a^2$ by $3\,a^m$.

[29] Take the fraction, $\dfrac{3\,ab}{4\,c}$.

If we multiply each term of the fraction by $a^{-1}b^{-1}c^{-1}$, we shall have $\dfrac{3\,a^0b^0c^{-1}}{4\,a^{-1}b^{-1}c^0}$. This equals, by previous case, $\dfrac{3\,c^{-1}}{4\,a^{-1}b^{-1}}$. That is, we have changed the factors a and b

from numerator to denominator, and c from denominator to numerator, by changing the signs of the exponents.

Take a general case, as $\frac{a^m}{b^n}$, in which m and n are any exponents.

Multiplying each term of the fraction by $a^{-m}b^{-n}$, we have

$$\frac{a^0b^{-n}}{a^{-m}b^0}=\frac{b^{-n}}{a^{-m}}.$$

We conclude that,

A letter that is a factor of either divisor or dividend may be changed from one term to the other by changing the sign of its exponent.

Transpose the letters having negative exponents to the other term.

Divide:

1. $12\,ab^{-2}x^4$ by $2\,b^2c^5x$.
2. $12\,b^2c^4x^{-1}$ by $2\,a^{-2}b^2c^5$.
3. $24\,a^2x^2y$ by $6\,a^{-4}x^3yz$.
4. $16\,abc^{-2}$ by $4\,a^2b^{-2}c^2$.
5. $25\,axy^{-1}$ by $5\,axyz^{-2}$.
6. $32\,a^2x^{-3}y$ by $8\,a^{-3}x^3y^3z^3$.
7. $20\,a^mx^ny^{-1}$ by $4\,ax^{-n}y^{n-1}$.
8. $27\,a^3b^{-2}c^{-1}$ by $9\,a^nb^{-2}c$.

[30] Divide:

9. $15\,x^{\frac{1}{2}}y^{-\frac{1}{2}}z$ by $3\,x^{-\frac{1}{4}}yz^{-\frac{1}{2}}$.
10. $18\,a^{\frac{1}{4}}bc^{-1}$ by $6\,a^{-\frac{3}{4}}b^2c^{-2}$.

In the following, transfer the different letters from dividend to divisor, and *vice versa:*

1. $\frac{3\,a^2x}{by}=\frac{3\,b^{-1}y^{-1}}{a^{-2}x^{-1}}$.
2. $\frac{2\,a^{-1}b^2}{c^{-2}y}$.
3. $\frac{4\,a^{-m}b}{2\,c^{-n}d^{-2}}$.
4. $\frac{6\,c^md^{-m}}{2\,a^2b^{-2}}$.

THE PARENTHESIS

On pages 16, 38, and 51 we took up the subject of the parenthesis. We shall now discuss it more fully.

As we may sometimes have a parenthesis within a parenthesis, it is necessary to have several signs which are used for the same purpose.

They are: ( ) parenthesis, [ ] brackets, { } braces, —— vinculum. These are called **signs of aggregation.**

To show the uses of these various signs, let us simplify the following expressions without combining:

$$24-[30-\{10-(9-\overline{5+2})\}].$$

We remove the innermost sign first, which is the vinculum. It has the minus sign before it. Hence we have

$$24-[30-\{10-(9-5-2)\}].$$

The parenthesis is next. Removing it, we have

$$24-[30-\{10-9+5+2\}].$$

Removing the braces and brackets in order, we have

$$\begin{aligned}&24-[30-10+9-5-2]=\\&24-30+10-9+5+2=\\&24-39+17=\\&41-39=2.\end{aligned}$$

[31] Remove the signs of aggregation from the following expressions. Do not attempt to remove more than one sign at a time. Commence with the innermost sign and work outward:

1. $15-(10-9-\overline{7-4})$.
2. $16-(18-\overline{7+8}-\overline{5-12})$.
3. $3-[17-(34-20)-\overline{7+18}+18]$.

4. $3(3+\overline{7-4})-4(24-\overline{13+9})$.
5. $9-[3-(7+6)+14-\overline{12-9}]$.
6. $27-(13-\{17-\overline{32-19}\}+3)$.
7. $-\{-[-(-\overline{3+4})]\}$.
8. $-\{-3(9+7)+7(9-6)\}$.
9. $3-\{4-[5-(6-\overline{7-8})-9]-10\}$.
10. $3a-(4a-\overline{2a-b})$.
11. $2a-(2a+b-\overline{3a+4b})$.
12. $4y-(2y-\overline{4y+3y}-6y)$.
13. $9m-(3m+2m-\overline{4m-m})$.

[32] 14. $(3ax-3ay)-(2ax-\overline{4ay-4ax})$.
15. $\{3x-[2x-(x-y)]\}-[4x-(3x-\overline{2x-y})]$.
16. $(3ab+2b^2-4c)-(2b^2-\overline{6c+ab})$.
17. $x-y-(x+y-4)$.
18. $3y-(y+3x-\overline{2x-y})$.

Put each of the answers to the last three examples within a ( ) with the minus sign before it.

19. $3(a-b+c)-(a-4b-3c)+\{a-(\overline{3a-2c}-b)\}$.
20. $3x-\{2x-[4y-(7x-\overline{8x+4x}+4y)+y-7x]\}$.
21. $5a-[-3a-7(2a-2b)-16b]$.
22. $a-[2a-(3a-\overline{4a-5a})-6a]$.
23. $2a-\{-[-(-\overline{2a-3a})]\}$.

ORIGINAL WORK

Make up an example like one of the above and work it out.

FACTORING

[33] Review Factoring on pages 53, 54.

To separate a trinomial into two equal binomial factors.

Review Set I, page 92, and Set II, page 93.

$$(a+b)(a+b)=(a+b)^2=a^2+2\,ab+b^2.$$

$$(a-b)(a-b)=(a-b)^2=a^2-2\,ab+b^2.$$

What, then, are the factors of $a^2+2\,ab+b^2$? of $a^2-2\,ab+b^2$?

How do these two trinomials differ?

How do the two sets of factors differ?

What part of the trinomial determines the sign which connects the terms?

One of the two equal factors of a quantity is called its **square root.**

To take the square root of a quantity we take the quantity with its exponent divided by 2. (Why?)

The square root of a^4 is a^2, because $a^2\times a^2=a^{2+2}=a^4$.

The square root of a^{2m} is a^m, because $a^m\times a^m=a^{2m}$.

The sign of square root is the radical sign $\sqrt{\ }$. Thus, $\sqrt{4}=2$; $\sqrt{a^2}=a$.

PRINCIPLE. *A trinomial is a perfect square when one term is twice the product of the square roots of the other two.*

RULE. *To factor a square trinomial, take the square roots of the square terms and connect these roots by the sign of the other term. The result will be one of the equal factors.*

NOTE. Always see whether or not the trinomial is a perfect square.

Resolve the following trinomials that are perfect squares into two equal binomial factors:

1. $a^2 + b^2 - 2ab$.
2. $a^2 + x^2 - 2ax$.
3. $b^2 - 2bx + x^2$.
4. $x^2 - 2xy + y^2$.
5. $m^2 - 2mn - n^2$.
6. $b^2 + 6bc + 9c^2$.
7. $49 + 84 + 36$.
8. $a^2x^2 + 2axy + y^2$.
9. $a^2x^2 + 4ax + 4$.
10. $x^2y^2 + 8xyz + 16z^2$.

[34]

11. $a^2 - 10a + 25$.
12. $x^2 + 2x + 1$.
13. $9x^2 - 18xy + 9y^2$.
14. $144m^2 + 120mn + 25n^2$.

OPTIONAL WORK

1. $1 + 16a + 64a^2$.
2. $25 - 10x + x^2$.
3. $36a^4 + 60a^2b^2 + 25b^4$.
4. $9m^2n^2 - 36mn + 36$.
5. $x^4 - 14x^2y^2 + 49y^4$.
6. $9x^2 - 12xy + 4y^2$.

ORIGINAL WORK

Make up three examples under this case and present them to the class for solution. Do not make them by squaring a binomial; but write two square end terms and form the middle term from the two already written. Thus, write for the end terms "$9a^2$" and "$49b^2$"; the middle term must be twice the product of the square roots, or $2 \times 3a \times 7b$, or $42ab$. You may make it + or − as you choose. The square trinomials will be

$$9a^2 - 42ab + 49b^2, \text{ or } 9a^2 + 42ab + 49b^2.$$

[35] To factor a binomial which is the difference of two squares.

Review Set III, page 93.

$(a+b)(a-b)=a^2-b^2$. $(x+2)(x-2)=x^2-4$.

What, then, are the factors of a^2-b^2? of x^2-4?

RULE. *To factor a binomial which is the difference of two squares, find the square root of each square term and write the sum of these square roots for one factor and their difference for the other.*

Factor the following:

1. x^2-y^2.
2. m^2-n^2.
3. c^2-d^2.
4. a^2-4.
5. m^2-9.
6. c^4-d^2.
7. y^2-25.
8. $9-a^4$.
9. $25-36\,d^2$.
10. $a^2b^2-c^2d^2$.
11. $x^{2m}-y^{2m}$.
12. $100-49$.
13. $64-25$.
14. x^2-144.
15. $81\,y^2-81\,x^2$.

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16. $16\,a^2b^2-9\,m^2n^2$.
17. $1-z^2$.
18. y^4-1.
19. y^6-z^6.
20. a^8-b^8.
21. $144\,m^4-81\,n^2p^2$.
22. $121\,x^{2m}-4\,y^{4m}$.
23. $169\,a^6-196\,b^8$.
24. $324\,p^4-256\,p^2$.
25. $9\,x^{4m}y^{2m}-361\,y^{4m}$.
26. $16\,x^{2n}-36\,y^4$.
27. $625\,x^4-400\,y^6$.

ORIGINAL WORK

Write down the difference of two squares at random and find the factors. Make up five examples of this kind.

[37] On page 96 you learned how to write the product of two binomials having a common term; such as,

$$(x+5)(x+2)=x^2+7x+10.$$
$$(x-5)(x-2)=x^2-7x+10.$$
$$(x+5)(x-2)=x^2+3x-10.$$
$$(x-5)(x+2)=x^2-3x-10.$$

Trinomials of this form are called **quadratic trinomials.** Conversely, quadratic trinomials can be resolved into their binomial factors.

I. *The square root of the square term of a quadratic trinomial is the first term of each binomial.*

II. *The second terms are numbers whose product equals the third term and whose algebraic sum is the coefficient of the second term.*

Factor $x^2-7x+12$.

The first term of each factor is $\sqrt{x^2}$ or x.

The factors of 12 are 6×2, 12×1, and 3×4.

Since 12 is $+$, the signs of the second terms must be alike. (Why?) Since $7x$ is $-$, the signs of the second terms must be minus. The question is, What numbers are there whose product is 12 and whose sum is -7? -3 and -4 answer these conditions. Hence the factors are $(x-3)$ and $(x-4)$.

Factor $x^2-5x-24$.

The factors of 24 are 6×4, 8×3, 12×2, 24×1.

Since the third term is $-$, the second terms have unlike signs.

Since the second term of the trinomial is $-$, the negative quantity is the greater. Of the factors of 24, which fulfill these conditions?

It will be seen at a glance that 8 and 3 are the only factors whose difference is 5. The 8 is $-$, and the 3 $+$. Hence the factors are $(x-8)(x+3)$.

Inspect carefully the third term with its sign. If this sign is +, you know that both the signs of the second terms in the factors are the same as the sign of the second term in the trinomial. If it is −, you know the signs of the second terms in the factors are unlike: the larger factor of the third term in the trinomial has the same sign as its second term. Take those factors of the third term whose algebraic sum is the coefficient of the second term.

Finally, prove that your factors are correct by multiplying them together.

Factor:

1. $a^2 - 5a + 4$.
2. $b^2 + 6b + 8$.
3. $x^2 - 3x - 4$.
4. $y^2 - 2y - 15$.
5. $m^2 - 4m - 12$.
6. $x^2 + 7x + 10$.

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7. $c^2 - 13c + 12$.
8. $d^2 - 5d - 6$.
9. $y^2 + 2y - 24$.
10. $z^2 + 8z - 48$.
11. $y^2 - 5y - 14$.
12. $x^2 - 10x + 16$.
13. $x^2 - 13x + 40$.
14. $y^2 - 9y + 20$.
15. $a^2 - 3a - 18$.
16. $b^2 - 15b + 14$.

ORIGINAL WORK

Make up two examples under this case by assuming the factors and taking their product. Present them to the class for solution.

OPTIONAL WORK

Factor the following algebraic expressions:

1. $81x^2 - 18xy + y^2$.
2. $25a^2 + 40ab + 16b^2$.
3. $16x^4 + 32x^2y^2 + 16y^4$.
4. $x^6 - 2x^3y^3 + y^6$.
5. $4x^4 + 44x^2y^2 + 121y^4$.
6. $a^{2m} - 2a^mb^m + b^{2m}$.

7. $144\,m^2 - 312\,mn + 169\,n^2$.

8. $36\,a^2 - 36\,b^2$.

9. $144\,m^2 - 169\,n^2$.

10. $a^{2m} - b^{2m}$.

11. $x^{2m} - y^{2m}$.

12. $(x - y)^2 - (x + y)^2$.

See whether this gives the same result as squaring first and then reducing.

13. $(a + 3)^2 - (a - 3)^2$.

Work this as you did Example 12.

14. $x^2 - 21\,x - 72$.

15. $a^2 - 20\,x + 96$.

REDUCTION OF FRACTIONS

[39] Review the work on fractions with monomial denominators, pages 57 to 61.

As the same principles are involved, the same rules must apply to fractions with binomials for denominators.

[40] To reduce fractions to lowest terms.

Reduce the following fractions to lowest terms:
(Resolve each term into its factors, and cancel if possible.)

1. $$\frac{a^2 - 2\,ab + b^2}{a^2 - b^2} = \frac{(a - b)(a - b)}{(a + b)(a - b)} = \frac{a - b}{a + b}.$$

2. $\dfrac{x - y}{x^2 - y^2}$.

3. $\dfrac{x^2 + 2\,xy + y^2}{x + y}$.

4. $\dfrac{2\,ab}{a^2 + ab}$.

5. $\dfrac{ax - a}{ay - a}$.

6. $\dfrac{x^2 - 5\,x + 6}{x^2 - 3\,x}$.

7. $\dfrac{x^2 - 3\,x + 2}{ax - a}$.

8. $\dfrac{c^2 + c}{3\,bc + 3\,b}$.

9. $\dfrac{x^2 - y^2}{3\,x^2 + 3\,xy}$.

[41] To reduce fractions to entire or mixed quantities.

Reduce the following fractions to entire or mixed quantities. See page 62.

1. $\dfrac{a^2-2ab+b^2}{a-b}=a-b.$

2. $\dfrac{a^2+b^2}{a+b}.$ 5. $\dfrac{a^3+1}{a+1}.$ 8. $\dfrac{3a^2-2a+4}{a+4}.$

3. $\dfrac{a^2+b^2}{a-b}.$ 6. $\dfrac{a^2+3a+7}{a+1}.$ 9. $\dfrac{3ax+3ay}{x+y}.$

4. $\dfrac{a^2+1}{a-1}.$ 7. $\dfrac{ab+3b^2}{a+b}.$ 10. $\dfrac{5x^2+4x+5}{5x+1}.$

[42] To reduce mixed quantities to fractions.

Reduce the following mixed quantities to a fractional form:

1. $2a-\dfrac{3ab}{a+b}=\dfrac{2a(a+b)-3ab}{a+b}=\dfrac{2a^2+2ab-3ab}{a+b}$

$$=\frac{2a^2-ab}{a+b}.$$

2. $6y-\dfrac{4ay}{a-b}.$ 3. $\dfrac{2x-a}{a-1}+1.$ 4. $\dfrac{2x-1}{4}+x.$

5. $2x-2-\dfrac{3x^2-2}{2x-2}.$ 8. $\dfrac{3a}{3x-2a}+2.$

6. $x+y-\dfrac{x^2+y^2}{x+y}.$ 9. $\dfrac{b^2}{a+b}+a-b.$

7. $\dfrac{3x-2}{x-4}+x+1.$ 10. $a-b-\dfrac{2ab+b^2}{a-b}.$

ORIGINAL WORK

[43] Make up three examples under each of the above cases, and find the required result.

OPTIONAL WORK

Reduce the following to their lowest terms:

1. $\dfrac{a^3-b^3}{3a-3b}$. **2.** $\dfrac{x^4-2x^2y^2+y^4}{4x^2-4y^2}$. **3.** $\dfrac{x^2+3x-10}{2ax-4a}$.

Reduce the following to mixed quantities:

4. $\dfrac{3a^4+10a^2-12}{3a^2-2}$. **5.** $\dfrac{9a^2-7a+3}{3a-3}$. **6.** $\dfrac{x^3-x^2y+xy^2-2y^3}{x-y}$.

Reduce the following to the fractional form:

7. $3a-4x-\dfrac{8x^2-14ax}{2a-2x}$. **8.** $x^2-xy+y^2-\dfrac{x^3}{x+y}$.

9. $a^3+a^2x+ax^2+x^3+\dfrac{x^4}{a-x}$.

ADDITION AND SUBTRACTION OF FRACTIONS

[44] Review L. C. M. on pages 55 to 57.

Review Addition and Subtraction of Fractions on pages 65 to 67.

[45] When one of the fractions has a binomial denominator.

Add $\dfrac{2}{a}$ and $\dfrac{4}{a-1}$.

The L. C. M. $= a(a-1)$.

By multiplying both terms of the first fraction by $a-1$, and both terms of the second fraction by a, we have

$$\frac{2(a-1)}{a(a-1)}+\frac{4a}{a(a-1)}=\frac{2a-2+4a}{a(a-1)}=\frac{6a-2}{a^2-a}.$$

Solve the following:

1. $\dfrac{a-3}{x+4}+\dfrac{4}{x}.$

2. $\dfrac{x^2+y^2}{x-y}-\dfrac{x+y}{2}.$

3. $\dfrac{5}{a+x}-\dfrac{5}{a}.$

4. $\dfrac{x+y}{3}-\dfrac{x-y}{3}-\dfrac{2}{a+b}.$

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5. $\dfrac{1}{y}+\dfrac{y}{x^2-y^2}.$ 6. $\dfrac{a-5}{a+3}-\dfrac{a-1}{a}.$ 7. $\dfrac{x-1}{x^2-1}+\dfrac{x^2-x}{x^2}.$

Reduce the fractions to their lowest terms before adding:

8. $\dfrac{a}{y+3}+\dfrac{b}{4}+\dfrac{c}{2}.$

9. $\dfrac{3x}{x-3}+\dfrac{2x}{4}.$

10. $\dfrac{3}{x^2-1}+\dfrac{2}{x^2}.$

11. $\dfrac{x}{x^2-xy}-\dfrac{3a}{9a}.$

ORIGINAL WORK

Make up one example under addition and one under subtraction of fractions and find the results.

[47] Find the result of:

1. $\dfrac{a}{a+b}+\dfrac{ab}{a^2-b^2}.$

2. $\dfrac{x}{x^2-y^2}-\dfrac{1}{x-y}.$

3. $\dfrac{8}{x^2-9}-\dfrac{4}{x+3}.$

4. $\dfrac{x+1}{x-1}-\dfrac{x-1}{x+1}.$

5. $\dfrac{4}{6x-3}+\dfrac{2}{2x-1}.$

6. $\dfrac{x+y}{x-y}-\dfrac{x-y}{x+y}-\dfrac{3xy}{x^2-y^2}.$

7. $\dfrac{2}{x+3}-\dfrac{1}{x+2}-\dfrac{x}{x^2+5x+6}.$

[48] 8. $\dfrac{2x+1}{2x-1} - \dfrac{4x(x+1)}{4x^2-1}$.

9. $\dfrac{a+1}{a+3} - \dfrac{a-2}{a-4}$.

10. $\dfrac{1}{a-x} + \dfrac{1}{a+x} - \dfrac{x^2+2a-a^2}{a^2-x^2}$.

MULTIPLICATION OF FRACTIONS

Review Multiplication of Fractions on pages 67 and 68, taking the optionals.

RULE. *When one of the fractions has a binomial denominator, multiply the numerators together for the new numerator and the denominators together for the new denominator, canceling when possible.*

An entire quantity may be treated as a fraction whose denominator is 1.

Multiply $\dfrac{x+3}{x-5}$ by $\dfrac{x^2-25}{3}$.

Canceling $x-5$, $= \dfrac{x+3}{\cancel{x-5}} \times \dfrac{\overset{x+5}{\cancel{x^2-25}}}{3}$

$= \dfrac{(x+3)(x+5)}{3}$

$= \dfrac{x^2+8x+15}{3}$.

DIVISION OF FRACTIONS

[49] Review Division of Fractions on pages 69 to 71.

RULE. *When one of the fractions has a binomial denominator, invert the divisor and proceed as in multiplication.*

Cancel whenever possible. An entire quantity may be treated as a fraction with 1 for a denominator.

Divide $\dfrac{1}{a^2 - x^2}$ by $\dfrac{1}{a - x}$.

$$= \frac{1}{a^2 - x^2} \times \frac{a - x}{1}.$$

Canceling $a - x$, $$= \frac{1}{\underset{a+x}{\cancel{a^2 - x^2}}} \times \frac{\cancel{a - x}}{1}$$

$$= \frac{1}{a + x}.$$

Divide $\dfrac{x - 3}{x - 2}$ by $x^2 - 9$.

$$= \frac{x - 3}{x - 2} \times \frac{1}{x^2 - 9}.$$

Canceling $x - 3$, $$= \frac{\cancel{x - 3}}{x - 2} \times \frac{1}{\underset{x+3}{\cancel{x^2 - 9}}}.$$

Multiplying, $$= \frac{1}{(x - 2)(x + 3)}$$

$$= \frac{1}{x^2 + x - 6}.$$

EXAMPLES IN MULTIPLICATION AND DIVISION OF FRACTIONS

1. $\dfrac{a}{a-x} \times \dfrac{a^2-x^2}{a}$.

2. $\dfrac{a-x}{a^2} \times \dfrac{a^3}{a^2-x^2}$.

3. $\dfrac{3a-3}{a-2} \times \dfrac{a^2-4}{3}$.

4. $\dfrac{x^2}{x^2-16} \times \dfrac{x-4}{x}$.

5. $\dfrac{x+y}{x-y} \times \dfrac{x^2-y^2}{x}$.

6. $\dfrac{a-b}{3a} \times \dfrac{9a^2}{2a^2-2b^2}$.

7. $\dfrac{1+x}{1-x} \times \dfrac{1-x^2}{1+x^2}$.

8. $\dfrac{x-5}{x-4} \times \dfrac{x^2-16}{5}$.

9. $\dfrac{xy+y^2}{xy} \times \dfrac{x^2y}{x^2-y^2}$.

10. $\dfrac{x}{a-x} \times \dfrac{y(a+x)}{a^2-x^2} \times \dfrac{(a-x)^2}{y^2}$.

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11. $\dfrac{1-x}{c} \div \dfrac{1-x^2}{c^2}$.

12. $\dfrac{a+y}{b^2} \div \dfrac{a^2+ay}{abc}$.

13. $\dfrac{8x^2y}{3x-6} \div \dfrac{2x^2yz}{x^2-4}$.

14. $\dfrac{x^2-z^2}{3ax} \div x+z$.

ORIGINAL WORK

Make up two examples under multiplication and two under division of fractions, one of the fractions in each having a binomial denominator, and find the results.

OPTIONAL WORK

1. $\left(a-\dfrac{b^2}{a}\right) \times \left(\dfrac{a}{b}+\dfrac{b}{a}\right)$.

2. $\left(\dfrac{a}{b}-2\right) \times \left(1-\dfrac{2a}{3y}\right)$.

3. $\dfrac{a^m}{b^n} \times \dfrac{a^n}{b^m}$.

4. $\dfrac{y-1}{x} \div \left(1+\dfrac{x}{1-x}\right)$.

5. $\left(1+\dfrac{1}{x}\right) \div \left(1-\dfrac{1}{x}\right)$.

6. $\left(\dfrac{a^2}{b}-b\right) \div \left(\dfrac{a}{b^2}+\dfrac{b}{a^2}\right)$.

FRACTIONAL EQUATIONS

[51] Review Equations, pages 72 to 80. We shall now consider fractional equations having a fraction with a binomial denominator.

Find the value of x in the following equation:

$$\frac{3}{x+2}+\frac{4x+1}{4}=\frac{5x+2}{5}.$$

It is generally better to clear of fractions with reference to the monomial denominators first, leaving the binomial denominator unchanged. You will see that this is much easier.

Multiplying the equation by the L. C. M. of 4 and 5, or 20, we have

$$\frac{60}{x+2}+20x+5=20x+8.$$

We can cancel $20x$ in each member. Transposing and combining, we have a simple form of a fractional equation.

$$\frac{60}{x+2}=3.$$

Clearing of fractions, $\quad 60=3x+6.$

Transposing, $\quad -3x=-54,$

Dividing by -3, $\quad x=18.$

VERIFICATION

$$\frac{3}{18+2}+\frac{72+1}{4}=\frac{90+2}{5}.$$

$$\frac{3}{20}+\frac{73}{4}=\frac{92}{5}.$$

$$\frac{3}{20}+\frac{365}{20}=\frac{368}{20}.$$

$$\frac{368}{20}=\frac{368}{20}.$$

[52] Find the unknown quantity in each of the following equations. Verify.

1. $\frac{x-5}{2}+\frac{2x-1}{3x+2}=\frac{5x-1}{10}-\frac{7}{5}.$

2. $\frac{6x+4}{15}+\frac{3x-2}{5x+6}=\frac{2x+3}{5}.$

3. $\frac{3x-3}{9}-\frac{x+6}{3x-4}=\frac{x-4}{3}.$

4. $\frac{7x+16}{21}-\frac{x+8}{4x-11}=\frac{x}{3}.$

5. $\frac{7x-29}{5x-12}-\frac{8x+19}{18}=-\frac{4x+3}{9}.$

OPTIONAL WORK

Find the value of the unknown quantity in each of the following equations:

1. $\frac{6x+13}{15}-\frac{2x}{5}=\frac{3x+5}{5x-25}.$

2. $\frac{7x-28}{5x-12}=\frac{8x+12}{18}-\frac{4x-3}{9}.$

3. $ab+b=bx+a.$

4. $\frac{1-x}{1+x}=1-\frac{1}{a}.$

5. $\frac{mx}{n}-\frac{a}{b}=\frac{n}{m}-\frac{bx}{a}.$

6. $\frac{x}{8}+\frac{5x}{4x+8}=\frac{11+x}{8}-\frac{3}{4}.$

PROPORTION

[53] The use of algebra in solving examples by proportion is most valuable.

Ratio is the relation with respect to magnitude which one quantity bears to another, and is found by dividing the first by the second. Thus the ratio of 8 to 4 is $\frac{8}{4}$, of a to b is $\frac{a}{b}$, etc.

The quantities that are to be compared must be of the same kind of units. Thus, there can be no ratio between 7 lb. and 14 yd., \$6 and 12 bu., 2 ft. and 6 yd., \$3 and 9 dimes.

The first term of a ratio is called the **antecedent**; the second, the **consequent**.

The sign of ratio is the colon (:). This is the division sign ($\div$) with the horizontal line omitted.

Proportion is the equality of ratios. It is, therefore, an equation.

The sign of proportion is the sign of equality (=), or the double colon (::).

Can we make a proportion or an equation of $6:3$ and $8:2$? of $5:2$ and $10:4$? Explain.

$5:10=6:12$ is a true proportion. Why?

In a proportion the first and last terms are called the **extremes**; the second and third terms, the **means**. The first and third terms are called the **antecedents**; and the second and fourth terms, the **consequents**.

Ratio is generally expressed in the form of a fraction. Hence a proportion is a fractional equation. Thus, $\frac{8}{4}=\frac{12}{6}$ is a proportion.

In the proportion $a : b = c : d$, or, as it is frequently written, $\frac{a}{b} = \frac{c}{d}$,

$\frac{a}{b}$ is the first ratio.

$\frac{c}{d}$ is the second ratio.

a and d are the extremes.

b and c are the means.

a and c are the antecedents.

b and d are the consequents.

Take the proportion $a : b = c : d$.

$$\frac{a}{b} = \frac{c}{d}.$$

Clearing of fractions, $$ad = bc.$$

$$a = \frac{bc}{d}.$$

$$b = \frac{ad}{c}.$$

PRINCIPLES. *In every proportion the product of the extremes equals the product of the means.*

Hence, either extreme equals the product of the means divided by the other extreme, and

Either mean equals the product of the extremes divided by the other mean.

To convert a proportion into an equation, place the product of the extremes equal to the product of the means.

[54] Convert the following proportions into equations and find the value of the unknown quantity. Then rewrite the proportion, using the value of the unknown quantity.

1. $3 : 5 = x : 400.$
2. $x : 9 = 7 : 21.$
3. $7 : x = 9 : 81.$
4. $36 : 54 = 30 : x.$
5. $x : b = c : d.$
6. $x + 3 : x + 8 = 10 : 12.$
7. $x - 6 : x + 4 = 40 : 60.$
8. $\frac{x}{4} : 8 = 12 : 16.$
9. $8 : 20 = x - 3 : x + 6.$
10. $x - 1 : 50 = 10 : 100.$

OPTIONAL WORK

1. If the antecedent is $8a^4c^2$, and the ratio is $4a^2$, what is the consequent?

2. If $a - x : a + x = 6 : 9$, what is the ratio of x to a? That is, find the value of the fraction $\frac{x}{a}$.

3. If $a : b = c : d$, prove that $a : c = b : d$.

4. If $\frac{a}{b} = \frac{c}{d}$, prove that $\frac{b}{a} = \frac{d}{c}$.

5. If $a : b = c : d$, is $b : a = d : c$? (Why?)

PROBLEMS

1. If the interest on \$140 for 1 year 6 months, at 4%, is \$8.40, what is the interest of \$249 for the same time and rate?

Let x = required interest.

Then $140 : 249 = 8.40 : x.$

$$140x = 2091.60.$$

$$x = 14.94.$$

Interest = \$14.94.

2. If the interest of $100 for 2 years at 6% is $12, what is the interest of $145 for the same time at 10%?

(Examine carefully the conditions so that in forming your proportion the ratios may be equal. If the first term is larger than the second, the third must be larger than the fourth, and *vice versa.*

3. What is that number to which if 4 is added and from which if 4 is subtracted, the results will have a ratio of 12 to 9?

[55] 4. What number is that which if it is doubled and if 30 is added to the number, the two results will be in the proportion of 10 to 15?

5. If 40 bushels of wheat cost as much as 60 bushels of rye, how many bushels of rye will cost as much as 100 bushels of wheat?

6. If 4 men can do as much work a week as 9 boys, how many men will it take to do as much work as 63 boys?

7. If an agent received $240 by selling a farm for $4000, how much would he have received at the same rate if he had sold it for $5000?

8. If 45 men consume a quantity of provisions in 32 days, how long would it last 80 men?

9. If a man saves $5200 in 8 years, how long will it take him to save $8450?

10. If 12 horses eat $30 worth of oats in five days, how much will it cost to feed 150 horses the same time?

11. If 12 men can do a piece of work in 33 days, how long will it take 18 men?

EQUATIONS WITH TWO UNKNOWN QUANTITIES

[56] If the sum of two numbers is 20, what are the numbers? How many unknown quantities are involved in this problem? If we call x and y the unknown quantities, we have an equation of $x + y = 20$.

Can we tell from this equation what the numbers are?

If $x = 4$, what is the value of y? Give other values to x and find y. How many positive integral numbers fulfill these conditions?

If we impose another condition and say, "What two numbers have 20 for their sum and 4 for their difference," do you think as many numbers fulfill the two conditions as fulfilled the one condition?

If x and y represent the numbers, since their sum is 20, and their difference is 4, we have

	(1)	$x + y = 20$
	(2)	$x - y = 4$
Adding equals to equals, we have	(3)	$2x = 24$
	(4)	$x = 12.$
Substituting value of x in (2),	(5)	$12 - y = 4.$
	(6)	$y = 8.$

We see that 12 and 8 fill the conditions imposed, and that no other numbers will.

In the equations $x + y = 10$ and $x + y = 12$, can x and y have the same values in each?

But if $$x + y = 10,$$
and $$2x + y = 16,$$

we can substitute 6 for x and 4 for y in each equation, and the equation will be satisfied; that is, the same unknown

quantities have the same value in each equation. They are called **simultaneous equations.**

The process of deducing a single equation containing one unknown quantity from two equations containing two unknown quantities is called **elimination.**

ELIMINATION BY ADDITION OR SUBTRACTION

Find the value of x and y in the following simultaneous equations:

(1) $$2x + y = 18,$$

(2) $$x - 2y = 4.$$

It is evident that to add or subtract the equations as they stand will not eliminate either unknown quantity. If we multiply the first equation by 2, we shall have

(3) $$4x + 2y = 36.$$

The coefficients of y are now alike.

If we add equation (3) to equation (2), the $2y$ in one will cancel the $2y$ in the other.

$$\begin{array}{lr} (3) & 4x + 2y = 36 \\ (2) & \underline{x - 2y = \ \ 4} \\ (4) & 5x = 40 \\ (5) & x = 8. \end{array}$$

If we now substitute the value of x in either equation, we shall be able to find the value of y. Substituting in the second equation, we have

$$\begin{array}{lr} (6) & 8 - 2y = 4. \\ (7) & -2y = -4. \\ (8) & y = 2. \\ (5) & x = 8. \end{array}$$

VERIFICATION

In (1) $2 \times 8 + 2 = 18.$

$16 + 2 = 18.$

In (2) $8 - 2 \times 2 = 4.$

$8 - 4 = 4.$

We could have eliminated the x first; as,

(1)	$2x + y = 18,$
(2)	$x - 2y = 4.$
(3)	$2x - 4y = 8$, multiplying second equation by 2.
(1)	$\underline{2x + \quad y = 18}$, bringing down (1).
(4)	$5y = 10$, subtracting (3) from (1).
(5)	$y = 2.$
(6)	$2x + 2 = 18$, substituting value of y in (1).
(7)	$2x = 16$, transposing and combining.
(8)	$x = 8.$
(5)	$y = 2.$

RULE. *To eliminate by addition or subtraction, multiply either equation, if necessary, by such a quantity as will make the coefficients of the same unknown quantity in the equations equal.*

If the signs of the equal coefficients are alike, subtract the equations; if unlike, add them.

From the resulting equation find the value of the unknown quantity. Substitute this value in one of the original equations for the other unknown quantity.

[57] Find the value of the unknown quantities. Verify:

1. $\begin{cases} 3x + 2y = 16, \\ 4x + 4y = 28. \end{cases}$

2. $\begin{cases} 2x + y = 12, \\ 3x + 5y = 39. \end{cases}$

3. $\begin{cases} x + y = 5, \\ 3x + 2y = 11. \end{cases}$

4. $\begin{cases} 4x + 3y = 43, \\ 8x - y = 23. \end{cases}$

5. $\begin{cases} 6x - y = 2, \\ 2x + 4y = 44. \end{cases}$

6. $\begin{cases} 3x + 3y = 36, \\ 4x - y = 13. \end{cases}$

7. $\begin{cases} 4x + 2y = 36, \\ 5x + 4y = 57. \end{cases}$

8. $\begin{cases} 3x + 5y = 82, \\ 4x - 2y = 14. \end{cases}$

(In Ex. 8, if the first equation is multiplied by 4 and the second by 3, 12 will be the coefficient for x in each.)

[58]

9. $\begin{cases} 6x - 5y = 11, \\ 5x - 4y = 10. \end{cases}$

10. $\begin{cases} 5y - 2x = 42, \\ 3y + 2x = 54. \end{cases}$

11. $\begin{cases} 7x + 9y = 51, \\ 11y - 2x = -1. \end{cases}$

12. $\begin{cases} 3x + 2y = 24, \\ 2y - 4x = 10. \end{cases}$

13. $\begin{cases} 5x + 2y = -2, \\ 2x - 5y = -24. \end{cases}$

14. $\begin{cases} 3z + 2y = 30, \\ 6z - 2y = 6. \end{cases}$

15. $\begin{cases} 3x - 2z = 17, \\ 9x + 2z = 91. \end{cases}$

16. $\begin{cases} 4x + 3y = 70, \\ 3x - 2y = 95. \end{cases}$

[59] In fractional equations, we may clear of fractions before eliminating, or we may eliminate the fraction. The latter method is frequently more simple. Proceed as follows:

(1) $\frac{x}{2} + \frac{y}{4} = 4,$

(2) $\frac{5x}{4} - \frac{y}{2} = 1.$

Dividing (2) by 2 by multiplying the denominators of the fractions, that is, multiplying the equation by $\frac{1}{2}$, we have

(3) $\frac{5x}{8} - \frac{y}{4} = \frac{1}{2}.$

Adding (1) and (3), (4) $\frac{x}{2} + \frac{5x}{8} = \frac{9}{2}.$

Clearing of fractions, (5) $4x + 5x = 36.$

(6) $9x = 36.$

(7) $x = 4.$

Substituting in (1), (8) $\frac{4}{2} + \frac{y}{4} = 4.$

Transposing and combining,

(9) $\frac{y}{4} = 2.$

Clearing of fraction, (10) $y = 8.$

(7) $x = 4.$

VERIFICATION

(1) $\frac{4}{2} + \frac{8}{4} = 4.$

$2 + 2 = 4.$

(2) $\frac{5 \times 4}{4} - \frac{8}{2} = 1.$

$5 - 4 = 1.$

We could have eliminated the x by multiplying the first equation by $\frac{5}{2}$, since that would change the coefficient of x to $\frac{5}{4}$. Try it.

Find the values of x and y in the following equations in two ways; first by clearing of fractions; second by eliminating the fraction:

17. $$\begin{cases} \frac{x}{3}+\frac{y}{4}=4, \\ \frac{2x}{3}-\frac{y}{2}=0. \end{cases}$$

18. $$\begin{cases} \frac{x}{3}+\frac{y}{2}=20, \\ \frac{x}{2}-\frac{y}{3}=4. \end{cases}$$

19. $$\begin{cases} \frac{x}{9}+\frac{y}{3}=12, \\ \frac{x}{6}-\frac{y}{2}=-6. \end{cases}$$

20. $$\begin{cases} \frac{5x}{6}-\frac{y}{12}=11, \\ \frac{x}{2}+\frac{y}{3}=25. \end{cases}$$

ORIGINAL WORK

[60] Make up five sets of equations containing two unknown quantities and integral terms. Present them for class solutions. Prove them correct by verification.

TWO OTHER METHODS OF ELIMINATION

[61] There are two other methods of eliminating the unknown quantity, which, in special cases, it may be expedient to use. These methods are called "Elimination by Comparison," and "Elimination by Substitution." In the first we "compare," or place equal, the same values of the unknown quantities; in the second we find the value of the unknown quantity in one equation and "substitute" it in the other. Elimination by Addition and Subtraction is, however, the method generally used.

ELIMINATION BY COMPARISON

Given (1) $2x + y = 16,$

(2) $4x + y = 28.$

From (1), (3) $y = 16 - 2x.$

From (2), (4) $y = 28 - 4x.$

Placing the values of y equal to each other,

(5) $16 - 2x = 28 - 4x.$

Transposing and combining,

(6) $4x - 2x = 28 - 16.$

(7) $2x = 12.$

(8) $x = 6.$

Substituting in (3), (9) $y = 16 - 12.$

(10) $y = 4.$

RULE. *Find the value of the same unknown quantity in each equation. Place these values equal to each other, and solve the resulting equation. Substitute this value in one of the given equations to find the value of the other unknown quantity.*

Solve the following equations by comparison and verify:

1. $\begin{cases} x + y = 15, \\ 2x + y = 20. \end{cases}$

2. $\begin{cases} 4x + 2y = 40, \\ 3x + y = 26. \end{cases}$

3. $\begin{cases} x + 2y = 18, \\ 3x + 5y = 47. \end{cases}$

4. $\begin{cases} 4x - 2y = 32, \\ 3x - y = 28. \end{cases}$

5. $\begin{cases} 3x + 2y = 70, \\ 2x + 3y = 75. \end{cases}$

6. $\begin{cases} 10x + 12y = 22, \\ 5x - 2y = 3. \end{cases}$

ORIGINAL WORK

Make up two sets of simultaneous equations containing two unknown quantities, and propose them for class solution.

OPTIONAL WORK

1. $\begin{cases} 6x + 3y = 18, \\ 4x - 2y = 20. \end{cases}$

2. $\begin{cases} 8x + 6y = 28, \\ 4x + 5y = 22. \end{cases}$

3. $\begin{cases} 3x = 5y, \\ 5(x - 4) = 6(y + 6). \end{cases}$

4. $\begin{cases} \frac{1}{2}x + \frac{1}{3}y = 7, \\ 2x + 3y = 43. \end{cases}$

ELIMINATION BY SUBSTITUTION

[62] Given as before in Comparison,

$$(1)\quad 2x + y = 16,$$
$$(2)\quad 4x + y = 28.$$

Finding value of y in (1),	(3)	$y = 16 - 2x.$
Substituting in (2),	(4)	$4x + (16 - 2x) = 28.$
	(5)	$4x + 16 - 2x = 28.$
	(6)	$2x = 12.$
	(7)	$x = 6.$
Substituting in (3),	(8)	$y = 16 - 12.$
	(9)	$y = 4.$

RULE. *Find the value of one of the unknown quantities in either equation. Substitute this value in the other equation, and solve.*

Solve the following equations by substitution and verify:

1. $\begin{cases} 2x + y = 14, \\ x + 2y = 16. \end{cases}$

2. $\begin{cases} 3x + 2y = 36, \\ x + 3y = 33. \end{cases}$

3. $\begin{cases} 2x + y = 23, \\ 7x + 2y = 67. \end{cases}$

4. $\begin{cases} 4x + 11y = 71, \\ 3x + 2y = 22. \end{cases}$

5. $\begin{cases} 7x - y = 11, \\ 9x - 2y = 7. \end{cases}$

6. $\begin{cases} 2y - 7x = 10, \\ 12x - y = 12. \end{cases}$

ORIGINAL WORK

Make up two sets of simultaneous equations containing two unknown quantities, and present to class for solution.

OPTIONAL WORK

1. $\begin{cases} 9x - 2y = -1, \\ 4x + 2y = 40. \end{cases}$

2. $\begin{cases} 2x + 2y = 74, \\ 3y - 3x = 33. \end{cases}$

[**63**] Solve Examples **3**, **5**, and **6**, under each of the two cases just given, by addition and subtraction.

Which of the three methods of elimination do you think the best? Why?

[**64**] Solve the following by any method you prefer:

1. $\begin{cases} 6x - 2y = -8, \\ 15x = 2y + 10. \end{cases}$

2. $\begin{cases} 3x - y = 0, \\ 12x + 2y = 126. \end{cases}$

3. $\begin{cases} 2x - 5y = 15, \\ 7x + 2y = 150. \end{cases}$

4. $\begin{cases} \dfrac{x}{2} + \dfrac{y}{3} = 12, \\ \dfrac{x}{3} + \dfrac{y}{9} = 6. \end{cases}$

5. $\begin{cases} \frac{1}{3}x + \frac{1}{2}y = 11, \\ \frac{1}{2}x + \frac{1}{3}y = 14. \end{cases}$

6. $\begin{cases} \dfrac{x+y}{6} + \dfrac{x-y}{3} = 10, \\ 2x - y = 36. \end{cases}$

7. $\begin{cases} \dfrac{x}{3} + \dfrac{y}{12} = \dfrac{8}{3}, \\ \dfrac{x}{9} + \dfrac{y}{3} = 3\frac{1}{3}. \end{cases}$

8. $\begin{cases} \dfrac{x+4}{x+y} + \dfrac{1}{2} = 1, \\ \dfrac{x-2}{y-x} + \dfrac{1}{2} = 1. \end{cases}$

TWO UNKNOWN QUANTITIES

[65] In solving simultaneous equations it is always necessary to have as many equations as there are unknown quantities. So in each of these problems you must always be able to state two equations involving different conditions. Many problems that can be solved by one unknown quantity can be, perhaps, more readily solved by two unknown quantities.

Thus, in Example 1, on page 81:

PROBLEMS

A farmer bought 4 lambs and 7 sheep for $54. If a sheep cost twice as much as a lamb, how much did each cost?

Let		x = price of each lamb.
		y = price of each sheep.
From first addition,	(1)	$4x + 7y = 54$
From second addition,	(2)	$y = 2x$
Bringing (1) down,	(1)	$4x + 7y = 54$
Transposing (2),	(3)	$-2x + y = 0.$
Multiplying (3) by 2,	(4)	$-4x + 2y = 0.$
Adding (4) and (1),	(5)	$9y = 54.$
	(6)	$y = 6$, price of each sheep.
Substituting y in (2),	(7)	$6 = 2x.$
		$x = 3$, price of each lamb.

Solve Problems 4, 5, 6, 13, 19, 32, and 33, on pages 81 to 84, by using two unknown quantities.

SOME OTHER PROBLEMS

[66] 1. A man bought at one time 4 sheep and 6 pigs for \$76. At another time he bought 3 sheep and 9 pigs at the same price, paying for them \$93. What was the price of each?

2. Find two numbers such that if 2 is added to the first and 4 is subtracted from the second, the sum of the results will be 20, and if the first is doubled and the second multiplied by 4, their sum will be 64.

3. A boy has 80¢ in his two pockets. If 5¢ is taken from the first pocket and put with the second, he will have an equal amount in each. How much money has he in each?

4. The sum of Mr. Jones's age and his wife's age is 84 years, and one half his age equals one half his wife's age plus 2 years. How old is each?

5. The daily wages of 6 men and 4 boys were \$16, and the wages of 10 men and 9 boys at the same rate were \$29. Find the daily wages of each.

[67] 6. A man owns two horses one of which cost \$100 more than the other. If \$64 is added to twice the cost of the better horse, it will equal 3 times the cost of the other. Find the cost of each horse.

7. The sum of the ages of a father and son is 120 years. If their ages are in the proportion of 3 to 2, what are their ages?

8. One number is to another as 4 to 5. If 15 is added to the smaller and 15 subtracted from the larger, they will be equal. What are the numbers?

9. If 1 is added to the numerator of a certain fraction, the value will be $\frac{3}{4}$; if 2 is added to the denominator, the value will be $\frac{1}{2}$. What is the fraction?

(Let $\frac{x}{y}$ = the fraction.)

10. If the numerator of a fraction is doubled, its value will be $\frac{3}{2}$. If 4 is added to the denominator, its value will be $\frac{3}{8}$. What is the fraction?

11. If 2 is added to the numerator of a fraction and 1 is added to the denominator, its value will be $\frac{5}{6}$; if 3 is subtracted from the numerator and 9 is subtracted from the denominator, its value will be 1. Find the fraction.

[68] **12.** A's money is to B's as 3 is to 4. If A should spend $10, he would then have $\frac{1}{2}$ as much as B. How much has each?

13. A son is $\frac{1}{4}$ as old as his father, but 20 years hence he will be $\frac{1}{2}$ as old. How old is each?

14. A man paid $\frac{3}{4}$ as much for a carriage as for a horse, and $\frac{1}{2}$ the price of the horse plus $\frac{1}{2}$ the price of the carriage equals $175. Find the price of each.

15. Two men form a partnership. A puts in $\frac{2}{3}$ as much as B. B puts in $5000 more than $\frac{1}{2}$ as much as A puts in. How much does each put in?

16. If to $\frac{1}{5}$ the sum of a man's age and his wife's age 12 is added, the sum will be 22; and if to $\frac{1}{2}$ the difference 13 is added, the result will be 15. What are their ages?

17. What two numbers are there to $\frac{1}{3}$ of whose sum if 5 is added the result will be the larger, and if 12 is added to the difference the sum will be 2 more than 3 times the smaller?

[69] **18.** A man divided one dollar among four children. To the first he gave 10 ¢ more than to the second, and to the third 10 ¢ more than he gave the fourth. If he gave the first twice as much as he gave the third, how much did he give to each?

19. A, B, C, and D invested \$6000 in speculation. A furnished twice as much as B; and C, twice as much as D. If B furnished \$400 more than D, how much did each contribute?

20. I have two purses containing \$190. If $\frac{1}{3}$ of the sum in the larger purse is added to $\frac{1}{2}$ the sum in the other, the amount will be \$30 more than the sum in the smaller purse. How much have I in each purse?

21. Find two numbers which are to each other as 3 to 2; and if 40 is added to each, their sums will be to each other as 7 to 6.

22. Divide 90 into two parts such that 4 times the larger shall exceed 3 times the less by 220.

23. The area of a rectangle equals the length multiplied by the width. If the length of a rectangle is increased 1 inch, the area will be increased 8 square inches. If the width is increased 2 inches, the area will be increased 18 square inches. Find the dimensions of the rectangle.

(Let x = length and y = width; then xy = area.)

24. I have a rectangular garden. If the length is increased by 9 feet and the width diminished by 1 foot, the area will be increased by 149 square feet. If the length is diminished by 1 foot and the width increased by 9 feet, the area will be increased by 249 square feet. Find the dimensions of the rectangle.

PERCENTAGE AND INTEREST

PROBLEMS

[70] 1. A wholesale merchant sold a lot of sugar at a gain of $50. If he had sold it so as to gain 10% more, he would have gained $300. Find the cost of the sugar and his gain per cent.

SUGGESTION. Let $x =$ cost of sugar; $\frac{y}{100} =$ the first rate, and $\frac{y+10}{100} =$ second rate. The equations will each contain xy. Eliminate xy from the equations and find the value of x.

2. A man sold a farm at a gain of a certain per cent and thereby cleared $1000. If he had sold it for $6250, he would have gained 5% more. What was the cost of the farm and what per cent did he gain?

3. If a certain principal in a certain time will amount to $480 at 5%, and to $448 at 3%, what are the principal and the time?

SUGGESTION. Let $x =$ principal, and $y =$ time in years.

The equations will be

$$\frac{5\,xy}{100} + x = 480. \qquad \frac{3\,xy}{100} + x = 448.$$

4. The amount of a certain principal for a certain time at 10% is $3700, and at 4% is $2680. Find the principal and time.

5. A sum of money placed at interest at a certain rate for 6 years will amount to $2080 and for 12 years, at the same rate, to $2560. Find the principal and the rate.

6. The amount of a certain principal at a certain rate for 4 years is $2560, and for 10 years is $3400. What are the principal and the rate?

QUADRATICS

[71] We have been considering equations in which the unknown quantities were of the first degree; that is, **simple equations.** If the unknown quantity in the equation is of the second power or degree, the equation is called an **equation of the second degree** or a **quadratic equation.**

Thus $x^2 = 49$ and $x^2 + 2x = 15$ are quadratic equations, as each contains the second power of the unknown quantity. $x^2 = 49$ is called a **pure quadratic** because it contains *only* the second power of the unknown quantity. $x^2 + 2x = 15$ is called an **affected quadratic** because it contains both the second power and the first power of the unknown quantity.

Every pure quadratic equation can be reduced to the form of $ax^2 = b$ in which a is any coefficient of x^2, and b the sum of the known quantities.

Solve:	(1)	$3x^2 - 16 = 32.$
Transposing and combining,	(2)	$3x^2 = 48.$
Dividing by 3,	(3)	$x^2 = 16.$
Extracting the square root of each member,	(4)	$x = \pm 4.$

If two quantities are equal, their square roots are equal. For each member being divided by itself, we have equals divided by equals; hence the results are equal.

The values of x are called **roots.** They are either + or −, since the square of either a positive or a negative quantity is positive; as, $(+4)^2 = (-4)^2 = 16$. If the square root is either + or −, the question will arise: why do we not write the ± before the x as well as before the 4, as,

$$\pm x = \pm 4?$$

If we do this, we shall have:

(1) $+x=+4$. (3) $+x=-4$.

(2) $-x=-4$. (4) $-x=+4$.

The first two reduce to $x=4$. The last two reduce to $x=-4$. Hence x has two values, $+4$ and -4, or ± 4; and it is unnecessary to write the $\pm$ signs on each side of the equation.

The square root of a quantity is indicated by the radical (root) sign $\sqrt{\ }$; thus $\sqrt{16}=\pm 4$, $\sqrt{a^2}=\pm a$.

A fractional exponent is also used to indicate the root of a quantity; thus $a^{\frac{1}{2}}=\sqrt{a}$.

There is no square root of a negative quantity, since the square of any quantity, positive or negative, is necessarily positive.

Find the values of the unknown quantity in each of the following equations. Verify:

[72]

1. $x^2-7=42$.
2. $x^2+24=105$.
3. $x^2-12=24$.
4. $3x^2-16=59$.
5. $5x^2-17=28$.
6. $\dfrac{5x^2}{8}+8=x^2+2$.
7. $6x^2-8-x^2=16+3x^2+48$.
8. $\dfrac{x(8+2x)}{10}=\dfrac{4x+25}{5}$.
9. $\dfrac{4x^2-4}{10}=32$.
10. $\dfrac{2}{3x^2}+\dfrac{5}{6x^2}=\dfrac{3}{8}$.

[73]

11. $\frac{x^2-4}{8}=\frac{x^2-16}{5}$.

12. $5(x+3)(x-3)=(2x+4)(2x-4)+71$.

13. $\frac{2x^2}{a}=8a$.

14. $10(x+4)(x-4)+110=(3x+1)(3x-1)$.

15. $\frac{x^2}{2}-3+\frac{5x^2}{12}=\frac{7}{24}-x^2+\frac{335}{24}$.

16. $12-x^2:\frac{x^2}{2}=100:25$.

PROBLEMS

[74] 1. If 27 is added to the square of a number, the sum will be 171. What is the number?

Solution

Let $x =$ the number.

$x^2 =$ the square of the number.

Then,
$$x^2+27=171.$$
$$x^2=144.$$
$$x=\pm 12.$$

2. When the square of a number is subtracted from 100, the result is 64. What is the number?

3. Find two numbers, such that one is 3 times as great as the other, and the difference of their squares is 72.

4. Find two numbers, one of which is twice the other, and the sum of whose squares is 80.

5. The length of a lot is to its width as 9 to 5, and the area is 4500 square feet. Find the dimensions.

6. The width of a lot is to its length as 2 to 6, and the area is 2700 square feet. Find the dimensions.

[75] 7. What number is that whose third part squared and added to 19 will give 100 for the sum?

8. Find two numbers such that their product is 64 and the quotient of the greater by the less is 4.

9. The difference of two numbers is 2, and the sum of their squares is 34. What are the numbers?

(It will be best not to let $x =$ one of the numbers, but a number halfway between the two. Then $x - 1$ and $x + 1$ will represent the numbers. Why? Try it letting $x =$ the number.)

10. The difference of two numbers is 4, and the sum of their squares is 170. What are the numbers?

(Let $x - 2$ and $x + 2$ represent the numbers.)

11. The difference of two numbers is 6, and the sum of their squares is 68. What are the numbers?

(Represent the numbers as you did in Problems **9** and **10**.)

12. A man has two sons, Frank and Edward. Frank is 9 years old. If the sum of Frank's age and Edward's age is multiplied by their difference, the product will be 65. How old is the younger?

13. Find two numbers, one of which is 4 times the other, and the sum of whose squares is 68.

OPTIONAL WORK

1. Two numbers are to each other as 3 to 5, and the sum of their squares is 136. What are the numbers?

SUGGESTION. Let $3x$ = one number and $5x$ the other.

2. The square of a certain number is equal to the difference of the squares of 25 and 15. What is the number?

3. If to 3 times the square of a certain number 8 is added, the sum will be 56. What is the number?

4. The sum of the ages of two boys multiplied by their difference equals 15. If the younger boy is 7 years old, what is the age of the elder?

5. The square of a certain number minus 9 is equal to one third its square plus 45. What is the number?

ANSWERS

FIRST HALF YEAR

Page 14. — **1.** 12. **2.** 12, 48. **3.** 25, 15. **4.** 20, 10.

Page 15. — **5.** 40. **6.** 11. **7.** 400 men, 200 women. **8.** Vest, \$7; coat, \$13. **1.** Vest, \$3; trousers, \$6; coat, \$12. **2.** 100, 300. **3.** Horse, \$160; harness, \$40. **4.** 90, 45, 15.

Page 22. — **1.** 16. **2.** 24. **3.** 0. **4.** 70. **5.** 150. **6.** 64. **7.** 6. **8.** 3. **9.** 140. **10.** 200. **11.** 32. **12.** 7. **13.** 128. **14.** 216. **15.** 27. **16.** 24. **17.** 200. **18.** 28. **19.** 81. **20.** 4. **21.** 1250. **22.** 7000. **23.** $\frac{1}{4}$. **24.** 1. **25.** 9. **26.** 10. **27.** 28. **28.** 18. **29.** 1. **30.** 10. **31.** 38. **32.** 1. **33.** 35. **34.** 49. **35.** 50. **36.** 25. **37.** 1. **38.** 16. **39.** 80. **40.** 5. **41.** 2. **42.** 0. **43.** 4. **44.** 2. **45.** 10. **46.** 40. **47.** 600.

Page 23. — **48.** 6. **49.** 4. **50.** 9. **51.** 1. **52.** 4. **53.** 3. **54.** 2. **55.** 3. **56.** 2. **57.** 8. **58.** 4. **59.** 2.

1. 351. **2.** 55. **3.** 1. **4.** ·1. **5.** 5. **6.** 15. **7.** 3. **8.** 1. **9.** $\frac{3}{4}$. **10.** $5\frac{4}{9}$.

Page 25. — **1.** $14a$. **2.** $15y$. **3.** $13xy$. **4.** $16ax$. **5.** $20y^2$. **6.** $21a^2$. **7.** $20y^2z^2$. **8.** $25abc$. **9.** $-18a$. **10.** $14(a+b)$. **11.** $-26xy$. **12.** $-19(x-y)$.

Page 26. — **1.** 10. **2.** -5. **3.** 14 dollars. **4.** -2¢. **5.** $3c$. **6.** $4ab$. **7.** $-2xy$. **8.** $2a^2b^2$. **9.** $-5xy^2$. **10.** $3(a+b)$. **11.** a. **12.** $7c$. **13.** $2xy$. **14.** $13a^2c^2$. **15.** $17ab^2$.

Page 27. — **1.** $9a+b+c$. **2.** $18b-12c-6d$. **3.** $a+b$. **4.** $a+b+c$. **5.** $a-b-c-d$. **6.** $2ax-az$. **7.** $-2a-4(b+c)$. **8.** $7a+3b+c$. **9.** $7ax-2by-d$. **10.** $3a^2-6c^2$. **11.** $4x^3-3z^3$. **12.** $6ab-6ab^2+5a^2b$.

Page 28. — **1.** $(a+b)c$. **2.** $(a+b+c)x$. **3.** $(a-b+c)y^2$. **4.** $(a+1)x$. **5.** $(a+b)(x+y)$. **6.** $(m-4)(a+b)$. **7.** $(4+a+b)(x-y)$. **8.** $(a^2+b^2+c^2)m$.

1. $a+b+c+d$. **2.** $a+2b+c$.

Page 29. — **3.** $2a+b$. **4.** $6d$. **5.** Suit, $2c+b$; for all, $4c+2b$. **6.** $4\frac{1}{2}a$. **7.** $a+d+16$. **8.** $a+b$. **9.** $a+b+c$. **10.** $m+n+p+r$. **11.** $2m+2n+s$.

Page 30. — **1.** $3ax-ay+7az$. **2.** $10a+7b+9c$. **3.** $3x$. **4.** $10a^2x-2by^2-6x^2y^2+2xy$. **5.** $5c^3+13c^2-5c+8$. **6.** $(a+b)x+(d+c)y+4xy$. **7.** $8(a+b)-(b+c)$. **8.** $4(x-y)+3(x+y)+3b$. **9.** $\frac{5}{8}ax+\frac{11}{12}z$. **10.** $\frac{13}{24}a^2+\frac{3}{8}b^2-\frac{7}{12}c^2+\frac{1}{12}d$.

Page 35. — **1.** $8ax$. **2.** $3ay-8b$. **3.** $6a-11b+2c$.

Page 36. — **4.** $3x-22$. **5.** $4(x+y)$. **6.** $-8ab-2b^2$. **7.** $4ab$. **8.** $-2a^2+2ax+5a^3x^3$. **9.** $-6ab^2-2a^3$. **10.** 7. **11.** $2x^3+4x^2-$

$10y^2-10$. **12.** $12x-3y-3$. **13.** $-3a^2b-3ab^2$. **14.** $160a-200a^2-2a^3-180m+9n$. **15.** m^2-n^2. **16.** a^2+b^2. **17.** $a^2+8ab-9m+22$. **18.** $6+3a+17b+10d-2c$. **19.** $3a^3+9ab^2-3a^2b+4b^3$. **20.** $5m^2$. **21.** $2a-6b-2c$.

Page 37. — **1.** $4a-b-c-3d$. **2.** $a+b+d-c-e-70$. **3.** $10-a$. **4.** x, and $5-x$. **5.** x, and $x+a$. **6.** $6a-6b$. **7.** $12c-12d$. **8.** $12m+12n-12a-12d$; $48m+48n-48a-48d$.

1. $(b-a)x$. **2.** $(d-c)y$. **3.** $(a-d)y$. **4.** $(a+b-c)y$. **5.** $6x-ay$.

Page 38. — **1.** $15a^2$. **2.** $4ab-4b^2$. **3.** $7-3a^2-2ab-b^2$. **4.** $4a$. **5.** $10b-a$. **6.** $10b-a$. **7.** $2b-7a$. **8.** $7a-2b$. **9.** $m+8n-6m^2$. **10.** $13ac-3bd+4a^2$.

Page 42. — **14.** $21a^2$. **15.** $-18b^2$. **16.** $6x^2$. **17.** $-12y^2$. **18.** $-12y^2$. **19.** $-28x^4$. **20.** $21a^2xy$. **21.** $6m^2n^2$. **22.** $-54axy^5$. **23.** $-36a^3b^3cd^2$. **24.** $16a^2bcxy$. **25.** $-150a^2c^4d^3$. **26.** $-14a^2b^2c^5$. **27.** $-4abxy$. **28.** $a^3b^4c^5d^6$. **29.** $-a^2yz^3$.

Page 43. — **30.** $28a^3p^{10}x^7z^{11}$. **31.** $3x^2y^5$. **32.** $16ab^2c^2x^3y^3z^3$. **33.** $-16c^7d^7$. **34.** $25x^{10}y^2z$. **35.** $-12a^2xy^{11}z$. **36.** $6(a+b)^5$. **37.** $-24(a-c)^4$. **38.** $-25(x+y)^6$. **39.** $12(x-y)^2$. **40.** $-28(a-b)^2$. **41.** $20(x+y)^3$.

Page 44. — **1.** $12a^2+18ac$. **2.** $-21c^3+6c^2x^2$. **3.** $-6ac^3d^2-8abc^2d^2-18ad^2c^2x$. **4.** $-2x^3y+4x^2y^2-2xy^3$. **5.** $-3a^3b^2+9a^2b^3-3ab^4$. **6.** $-a^2x^2yz+a^2xy^2z+a^2xyz^2$. **7.** $abm-cdm+efm$. **8.** $-xy+y^2+yz$. **9.** $-3a^2b^2-12ab^3+3b^4$. **10.** $ax-bx+cx$. **11.** $12a^9+8a^8b-4a^7c^3$. **12.** $-9a^2m^2x^2yz-6a^2m^2xy^2z+18a^2m^2xyz^2$. **13.** $9a^2x-12a^2y+6a^2z$. **14.** $8a^2bc-6ac^3+8acd^3$. **15.** $-18a^4b^3d^2-9a^5b^2d^4-3a^2b^3d^3$. **16.** $24a^2x^4y^4+12b^2x^3y^3+8x^3y^3$. **17.** $-9amn^2+6amn^3+12am^2n$. **18.** $28abx^2y-14abmxy-21a^2bx-28a^2b^2x$.

Page 45. — **19.** $-10a^3c^4+6a^4c^3-12a^4c^4$. **20.** $9x^4-9x^5-9x^6-9x^7$. **21.** $10a^3xz+6a^4yz+8a^3z^2-6a^2z$. **22.** $8a^2bx^2+16a^2bx^3y+32a^3b^2x$. **23.** $-a^3b^2c^2-a^3c^2e^2-a^3d^2e^2+a^3e^4$.

1. $6a^3x+8a^3y-4a^3z$. **2.** $15a^3y^3z^2-6a^4y^4z^2$. **3.** $9a(a+b)+6a(x+y)$. **4.** $8a+4a^2-2a^3$. **5.** $\frac{1}{9}c^3-\frac{1}{12}c^2-\frac{1}{6}c$. **6.** $a^{3m}+a^{2m}+a^{m+2}$. **7.** $a^{m+2}-a^4$. **8.** $y^{2n-1}-y^{2n}$. **9.** $4a^{m+2}+6a^{m+1}-10a^m$. **10.** $6(x+y)^4-4(x+y)^3$.

1. $40a$. **2.** ac.

Page 46. — **3.** $6b$. **4.** $a+ab$. **5.** $ab+4b$.

Page 49. — **1.** $4a$. **2.** $6ay$. **3.** $-3a^2x^2y$. **4.** $-4a$. **5.** $-6ay$. **6.** $-3a^2xy$. **7.** $20a^5y^5$. **8.** $8am^4z^4$. **9.** $-12m^9$. **10.** -8. **11.** $-4m^3n^2$. **12.** $-13d^2e^2$. **13.** $13a^2y$. **14.** $10bc$. **15.** $4a^3b^3c^3$.

Page 50. — **1.** $2-ay+3y^2$. **2.** $2y-3z-y^3z^3$. **3.** $9c^2-b-2abx^2$. **4.** $2a^3b-ab^3-2$. **5.** $3b-2b^2-1$. **6.** $-4ax-3-7ay$. **7.** $a^2-2ax+3xy-x^2$. **8.** $2ay+4a^2-3x^2+4m+2$. **9.** $3b+4ac-2a^2cx+8$. **10.** $2mn-1+6m^2ny-4m$. **11.** $1-6a^2ny-c-7a^3n$. **12.** $2-3xyz+8axy$.

Page 51. — **13.** $9a^3x^3 - 6x + 8a$. **14.** $-3a - 7a^2x - 3ax^2y^2 + 2y$. **15.** $2 + 3(a-y) + 6(a-y)^2$.

1. $-a^3 - 2a^5$. **2.** $x^3 + x^2z^2 + xyz^3 - yz$. **3.** $1 - 2(x+y) + 3(x+y)^2$. **4.** $1 + 4a + 6a^2$. **5.** a. **6.** $a - b$. **7.** $5ab - 3b^{-1}x - 1$.

Page 52. — **1.** $a - b$. **2.** $a - 4b$. **3.** $-a$. **4.** $13y - 12$. **5.** $5a + x + 6$. **6.** $4b$. **7.** $-2a$. **8.** $-4y$.

Page 54. — **1.** $4(a-b)$. **2.** $a(a-b)$. **3.** $a^3(b+4c)$. **4.** $7a(a+2b)$. **5.** $16xy(2 - 3x + yz)$. **6.** $18xy(4xy - 3)$. **7.** $9a^2b^2(1 - 2a)$. **8.** $8x(a^3y - 8xz)$. **9.** $14a(2bc - 3yz)$. **10.** $a^2b(7b + 3x)$. **11.** $3ax(a^2y + 3)$. **12.** $2by^2(2bx + 9a)$. **13.** $3xy^2(1 + 9y - y^2)$. **14.** $2x^2y^2(1 - 2z - 4a)$. **15.** $7a^2b(2a - abx - 3a^2b^2)$. **16.** $8a(4xy - 2ab^2 + 3a^2bx)$. **17.** $20x^2y(2 - 3xy^2 - 4x^2y^3)$. **18.** $9mn(mn - 3x)$. **19.** $13am(1 - 3m)$. **20.** $6a^2x^2(4a + 7xy)$. **21.** $3(ay - 2ay^2 + 4)$. **22.** $4a(b - 2an^2 + 2nx)$. **23.** $6ab(c - 2ab - 3a^2c)$. **24.** $7xy(xy - xz^2 - z)$. **25.** $5abm(2b - a + 3bm)$. **26.** $4a^2x^2y(4a - 5y)$. **27.** $5m^2xy^n(5x^2y^4 - 12m)$. **28.** $7a^2x^3y^3(7x^4 + 9y)$.

1. $9mn(mn - 3y + 9m^2n^2)$. **2.** $a^2b^2x^2(1 - 3abx - 6a^2b^2x^2)$. **3.** $2a^m(b + 2a^mc - 4a^md)$. **4.** $4x^{\frac{1}{2}}y^2(z - 4xy^2 - 7y)$. **5.** $16a^nb^m(2c^2 - 4a^nc^3 - 1)$.

Page 56. — **1.** $24a^3b^2x^2$. **2.** $36a^3b^3c^3$. **3.** $12a^3b^2x^2y^2$. **4.** $36x^3y^3z^3$. **5.** $20a^2c^2x^3$. **6.** $28a^2b^3c^2x^2$.

Page 57. — **1.** $40m^4n^3p$. **2.** $60x^4y^4z^4$. **3.** $36a^2b^2c^2d^2$. **4.** $32a^3bd^2x^2y^5$. **5.** $36a^3bc^4d$. **6.** $144a^2x^3y^2$. **7.** $90a^4b^3c^3$. **8.** $192b^2c^2x^2y^2$. **9.** $90x^4y^5$. **10.** $84a^2b^2c^2x^3$.

Page 60. — **1.** $\frac{a}{2x}$. **2.** $\frac{ay}{9bx}$. **3.** $\frac{4x}{3aby}$. **4.** $\frac{2b^3}{3ac^2d}$. **5.** $\frac{4a^4}{5bd}$. **6.** $\frac{4}{5xyz}$. **7.** $\frac{2}{3abc}$. **8.** $\frac{z^5}{5ax^2}$. **9.** $\frac{2}{3a^2z}$. **10.** $\frac{2b^4}{ay}$. **11.** $\frac{9x^2yz}{17ab}$. **12.** $\frac{14bc}{17a^4m}$.

1. $\frac{2x - 3a}{y}$. **2.** $\frac{3a - 3b}{b}$. **3.** $2ax - a$. **4.** $\frac{a^2 - 2b}{ab}$.

Page 61. — **5.** $\frac{1 - 3ax}{y}$. **6.** $\frac{2ab - 3a^2}{x}$. **7.** $a^2 - ac$. **8.** $\frac{5acy - 2acx}{xy}$. **9.** $\frac{2d - 3c}{c}$. **10.** $a - b$.

1. $\frac{2x^2}{5a^2y}$. **2.** $\frac{8y^5z}{9x^2}$. **3.** $\frac{1}{12z}$. **4.** $\frac{2a}{5}$. **5.** $\frac{5b^m}{9c^n}$. **6.** $\frac{3x^m}{8a}$. **7.** $\frac{4c}{7ab^2}$. **8.** $\frac{3c - 4d}{3a}$. **9.** $\frac{4 + 3d}{3}$. **10.** $\frac{a + b}{2}$. **11.** $\frac{2(x-y)^2}{3}$. **12.** $c^m - 2d^n$. **13.** $\frac{a + b}{3x}$. **14.** $\frac{3a}{x + y}$.

Page 62. — 1. $x+\frac{1}{x}$. 2. $3x-1+\frac{3}{x}$. 3. $x^2+x+1+\frac{1}{x}$. 4. $3+\frac{2b}{ax}$. 5. $5ab+\frac{3c}{5}$. 6. $a+\frac{b^3}{a^2}$. 7. $7a-4x$. 8. $4x-8+\frac{3}{5x}$.

Page 63. — 9. $3y+ab+1$. 10. $3x-6-\frac{7}{3x}$. 11. $2x^2-1-\frac{5}{8x}$. 12. $a^2+4+\frac{3}{4a^2}$. 13. $7a-6x-\frac{8}{9x}$. 14. $3+4xy-\frac{5}{xy}$.

1. $a-2b+\frac{1}{7a}$. 2. $1-3ac-\frac{x}{3ac}$. 3. $1-2a^mb-\frac{c}{2a^mb}$. 4. $3-9y-11a-5a^2y$. 5. $1-3xy+4x^2y^2-\frac{a}{xy}$.

Page 64. — 1. $\frac{23x}{9}$. 2. $\frac{26ab}{7}$. 3. $\frac{5ay}{3}$. 4. $\frac{17a^2}{5}$. 5. $\frac{a^3-b^3}{a}$. 6. $\frac{x^2-y^2}{x}$. 7. $\frac{a^4-b^4}{a^2}$. 8. $\frac{6a^2-8a+5b}{2}$. 9. $\frac{8a^2+3x+5}{2a}$. 10. $\frac{x^2+3ax-a^2}{x}$. 11. $\frac{24b+4x+c}{3}$. 12. $\frac{10mx+2n+n^2}{5x}$. 13. $\frac{31c-19a}{7}$. 14. $\frac{14-11a}{6}$.

Page 66. — 1. $\frac{31x}{24}$. 2. $\frac{77a^2}{45}$. 3. $\frac{15a^2}{4b}$. 4. $\frac{3}{b^2}$. 5. $\frac{7a-y}{12}$. 6. $\frac{17b-7a}{45}$. 7. $\frac{13a+23}{12}$. 8. $5a+\frac{4by-3x}{b^2}$. 9. $6x+\frac{3x+2}{4x}$. 10. $7y+\frac{9-4x}{3x^2}$. 11. $\frac{a^2c^2+b^2c^2+a^2b^2}{abc}$. 12. $\frac{7a+5x}{24}$. 13. $1-4a+\frac{5a+12}{6a}$. 14. $5x+8+\frac{5-2x}{3x}$. 15. $\frac{y-x}{20}$. 16. $\frac{19ax}{24}$. 17. $\frac{2}{c}$.

Page 67. — 1. $\frac{1-4x}{10}$. 2. $\frac{8y-5}{6x^2y^2}$. 3. 4. 4. a. 5. $2a-\frac{a}{20}$. 6. $\frac{2}{c}$. 7. 0.

Page 70. — 1. $24ab^2c^2d^2$. 2. $\frac{7ac}{d}$. 3. $15ab^2cxy$. 4. $\frac{10ax^2y}{9b}$. 5. $32a^2bcx$. 6. $\frac{3}{4c}$. 7. $\frac{xy}{6a^2}$. 8. $\frac{2xy}{15ab}$. 9. $\frac{9c}{32}$. 10. $\frac{ab}{28c^2}$.

1. $\frac{ab}{d}$. 2. $\frac{a-b}{bc}$. 3. $\frac{4ax^3}{3}$. 4. $\frac{m-n}{5d}$. 5. $\frac{ac+bc}{2d}$. 6. $\frac{y^2}{x^2}$. 7. $\frac{6bd}{ac}$. 8. $\frac{3ab}{2xy}$. 9. $\frac{5d^2}{3a}$. 10. $\frac{3ax-3bx}{2a}$.

[59] 1. $\frac{b^2c}{3d}$. 2. $\frac{cdxy}{2}$. 3. a^3x. 4. $\frac{3dm^2y^2}{4ab^2}$.

Page 71. — 5. $\frac{9x^3}{4yz^2}$. 6. $\frac{3n}{2b}$. 7. $\frac{x^3}{6ab^3}$. 8. $\frac{21a}{2c}$. 9. $\frac{9d^3}{98c^3y^3}$. 10. $\frac{bcmx}{3adny}$.

1. $\frac{a^2}{bc^2(a-b)}$. **2.** $\frac{ax+y}{a}$. **3.** $\frac{b}{c}$. **4.** $\frac{c+d}{4c}$. **5.** $\frac{3x+9y}{c}$. **6.** $\frac{a^{2m}}{y^{2m}}$.
7. $\frac{x^{m+n}}{y^{m+n}}$. **8.** 2. **9.** $\frac{b(x+y)}{6}$. **10.** $\frac{a+b}{2ab}$. **11.** $\frac{x-y}{2a^2b}$. **12.** 2.

Page 75.—**1.** 2. **2.** 5. **3.** 2. **4.** 3. **5.** 8. **6.** 8. **7.** 10. **8.** 1. **9.** 7.

Page 77.—**1.** 6. **2.** 6. **3.** 3. **4.** 7. **5.** 11. **6.** 1. **7.** 7. **8.** 4. **9.** 12. **10.** 4. **11.** 3. **12.** 4. **13.** 12. **14.** 10. **15.** 2. **16.** 8. **17.** 10. **18.** 3. **19.** 6. **20.** 35. **21.** 3. **22.** 6. **23.** 6. **24.** 8. **25.** 1. **26.** 5. **27.** 3. **28.** 3. **29.** 6. **30.** 10. **31.** 16. **32.** 10. **33.** 7. **34.** 9. **35.** 13. **36.** 2. **37.** −4. **38.** 5. **39.** 18. **40.** 3.

Page 78.—**41.** 40. **42.** 11. **43.** 12. **44.** 2. **45.** 10. **46.** 1. **47.** 4. **48.** 4. **49.** −5. **50.** 9. **51.** 4. **52.** 1. **53.** 3. **54.** 6. **55.** 15. **56.** 16. **57.** 12. **58.** 24. **59.** 20. **60.** 10. **61.** 12. **62.** 5. **63.** 10.

Page 79.—**67.** 12. **68.** 30. **69.** 6. **70.** 5. **71.** 28. **72.** 8. **73.** 2.

Page 80.—**74.** 28. **75.** 2. **76.** 4. **77.** 15. **78.** 66. **79.** 8. **80.** 2.

Page 81.—**2.** Algebra, 50¢; Dict., $4. **3.** 42 children, 84 women, 168 men. **4.** Frank, 20 yr.; sister, 10 yr. **5.** 24 and 96.

Page 82.—**6.** 30 and 45. **7.** 19 and 31. **8.** 50 and 100. **9.** 40 and 120. **10.** 40 and 120. **11.** 20 and 60. **12.** 50 and 100. **13.** Horse, $210; carriage, $150. **14.** A, $1800; B, $900; C, $2700. **15.** 18, 21, 24, 27.

Page 83.—**16.** Daughter, $10,000; son, $20,000; wife, $40,000. **17.** $70,000. **18.** $1200. **19.** 20 and 16. **20.** 60. **21.** 40 yr. **22.** 60 yr. **23.** 48 gal. **24.** 16. **25.** 4.

Page 84.—**26.** 20. **27.** Charlie, $120; Frank, $80. **28.** A, $2000; B, $1500; C, $1000. **29.** 17. **30.** 4. **31.** 48. **32.** Man, $2.50; son, $1.50. **33.** Son, $1; father, $1.50. **34.** $2000, $4000, $8000. **35.** 20, 21, 22. **36.** $16 and 16 quarters.

Page 85.—**1.** $1200. **2.** $2000. **3.** $240. **4.** $5000. **5.** $4.80. **6.** Cost, $480; gain, $180. **7.** $2200. **8.** $112. **9.** 25%. **10.** 20%.

Page 87.—**1.** $49. **2.** $38.88. **3.** $262.50. **4.** $117. **5.** $84. **6.** $2000.

Page 88.—**7.** $500. **8.** 5%. **9.** 2 yr. **10.** 4 yr.

1. $162. **2.** $110.50. **3.** $408. **4.** $400. **5.** $800. **6.** $1000. **7.** 6%. **8.** $4\frac{1}{2}$%. **9.** 5 yr. **10.** 4 yr. 6 mo.

SECOND HALF YEAR

Page 89.—**1.** $15+x-y$. **2.** $6a^2+14a+7b-10$. **3.** $4x^3+14x^2y+3xy^2+3y^3$. **4.** $2my^2+ny-3p$. **5.** $11a^{\frac{1}{2}}y+9$. **6.** $6a(x+y)$. **7.** $\frac{3}{4}ab+2\frac{2}{3}ay-\frac{1}{4}xy-\frac{1}{6}ax$. **8.** $a+\frac{1}{3}b-1\frac{1}{12}c$.

Page 91.—**1.** $a^2+14a+40$. **2.** $x^2-13x+22$. **3.** $x^2+xy-y-1$. **4.** x^2-9. **5.** $a^3-2a^2b+2ab^2-b^3$. **6.** $6a^4-96$. **7.** $3x^2+7xy+2y^2$.

8. $2a^5-4a^2-32a+16$. **9.** $6x^4-x^3-3x^2-17x-5$. **10.** a^3-48a^2+200. **11.** $4x^3y+6x^2y^2+x+2xy^3+y$. **12.** $6a^2x-ax+6a^2-a-x-1$. **13.** $2b^3-6b-4$. **14.** $x^2+x^2y-2xy^2-xy+y^3$. **15.** $-m^3+m^2+m^2n-mn^2-n^2+n^3$. **16.** x^3+1. **17.** $3a^2x^2+4a^2x+ax-4a-4$. **18.** $x^2+xy+ay-a^2$. **19.** $3x^2-8x-16$. **20.** $2a^2c^2+3ac^2-6c-8$.

Page 92. — **1.** $1-x^5$. **2.** $8a^3-27b^3$. **3.** $48x^3-92x^2y-40xy^2+100y^3$. **4.** $2x^4-2y^4$. **5.** $x^{2n}-y^{2n}$. **6.** $x^{n+1}+xy^n-x^ny-y^{n+1}$. **7.** $x^3-5x^2-46x-40$. **8.** $x+y$.

Page 100. — **1.** $a+b$. **2.** $x-y$. **3.** $x+3$. **4.** $x-4$. **5.** $a^2-2ab+b^2$. **6.** a^2-ab+b^2. **7.** $2x^2-3x+2$. **8.** $a^4+a^3b+a^2b^2+ab^3+b^4$. **9.** $x-1$. **10.** $x-3$. **11.** $a+7$. **12.** $b+2$. **13.** $m+2$. **14.** c^2-5. **15.** $dx-9$.

Page 101. — **16.** a^2+9. **17.** a^2c^2-4. **18.** x^3-3. **19.** x^2+y^2. **20.** $x^3+x^2y+xy^2+y^3$. **21.** $a^3-a^2b+ab^2-b^3+\frac{2b^4}{a+b}$. **22.** $8x^3-12x^2+18x-27$. **23.** $8x^3+12x^2+18x+27$. **24.** $4a+7$. **25.** $x-y-z$. **26.** $y+a$. **27.** $3a^2-2ab-b^2$.

1. $a+5b$. **2.** $4c-4d$. **3.** x^4-y^4. **4.** $a^2-4a+16$. **5.** a^2+ab-b^2. **6.** $3(m+n)+4$. **7.** $x^{\frac{1}{2}}-y^{\frac{1}{2}}$. **8.** a^2-a+1. **9.** $x^{\frac{1}{3}}+y^{\frac{1}{3}}$. **10.** a^m+b^m. **11.** a^2-a-1.

Page 106. — **1.** $\frac{6a^2x^3}{b^4c^5}$. **2.** $\frac{6a^2}{cx}$. **3.** $\frac{4a^6}{xz}$. **4.** $\frac{4b^3}{ac^4}$. **5.** $\frac{5z^2}{y^2}$. **6.** $\frac{4a^5}{x^6y^2z^3}$. **7.** $\frac{5a^mx^{2n}}{ay^n}$. **8.** $\frac{3a^3}{a^nc^2}$. **9.** $\frac{5x^{\frac{3}{4}}z^{\frac{3}{2}}}{y^{\frac{3}{2}}}$. **10.** $\frac{3ac}{b}$.

2. $\frac{2c^2y^{-2}}{ab^{-2}}$. **3.** $\frac{4c^nd^2}{2a^mb^{-1}}$. **4.** $\frac{6a^{-2}b^2}{2c^{-m}d^m}$.

Page 107. — **1.** 17. **2.** 6. **3.** 7.

Page 108. — **4.** 10. **5.** 8. **6.** 15. **7.** 7. **8.** 27. **9.** -2. **10.** $a-b$. **11.** $3a+3b$. **12.** $15y$. **13.** $7m$. **14.** $ay-3ax$. **15.** $-x$. **16.** $4ab+2c$. **17.** $4-2y$. **18.** $y-x$. **19.** $2b+8c$. **20.** $y-x$. **21.** $22a+2b$. **22.** $9a$. **23.** a.

Page 114. — **2.** $\frac{1}{x+y}$. **3.** $x+y$. **4.** $\frac{2b}{a+b}$. **5.** $\frac{x-1}{y-1}$. **6.** $\frac{x-2}{x}$. **7.** $\frac{x-2}{a}$. **8.** $\frac{c}{3b}$. **9.** $\frac{x-y}{3x}$.

Page 115. — **2.** $a-b+\frac{2b^2}{a+b}$. **3.** $a+b+\frac{2b^2}{a-b}$. **4.** $a+1+\frac{2}{a-1}$. **5.** a^2-a+1. **6.** $a+2+\frac{5}{a+1}$. **7.** $b+\frac{2b}{a+b}$. **8.** $3a-14+\frac{60}{a+4}$. **9.** $3a$. **10.** $x+\frac{3x+5}{5x+1}$.

2. $\frac{2ay-6by}{a-b}$. 3. $\frac{2x-1}{a-1}$. 4. $\frac{6x-1}{4}$. 5. $\frac{x^2-8x+6}{2x-2}$.
6. $\frac{2xy}{x+y}$. 7. $\frac{x^2-6}{x-4}$. 8. $\frac{6x-a}{3x-2a}$. 9. $\frac{a^2}{a+b}$. 10. $\frac{a^2-4ab}{a-b}$.

Page 116. — 1. $\frac{a^2+ab+b^2}{3}$. 2. $\frac{x^2-y^2}{4}$. 3. $\frac{x+5}{2a}$. 4. $a^2+4-\frac{4}{3a^2-2}$. 5. $3a+\frac{2a+3}{3a-3}$. 6. $x^2+y^2-\frac{y^3}{x-y}$. 7. $\frac{3a^2}{a-x}$.
8. $\frac{y^3}{x+y}$. 9. $\frac{a^4}{a-x}$.

Page 117. — 1. $\frac{ax+x+16}{x^2+4x}$. 2. $\frac{x^3+3y^2}{2(x-y)}$. 3. $-\frac{5x}{a^2+x^2}$.
4. $\frac{2ay+2by-6}{3(a+b)}$. 5. $\frac{x^3}{x^2y-y^3}$. 6. $\frac{3-7a}{a^2+3a}$. 7. $\frac{x^2+x-1}{x(x+1)}$.
8. $\frac{4a+3b+by+2cy+6c}{4(y+3)}$. 9. $\frac{2x^2+6x}{4(x-3)}$. 10. $\frac{5x-2}{x^2(x^2-1)}$.
11. $\frac{3-x+y}{3(x-y)}$.
1. $\frac{a^2}{a^2-b^2}$. 2. $\frac{y}{y^2-x^2}$. 3. $\frac{20-4x}{x^2-9}$. 4. $\frac{4x}{x^2-1}$. 5. $\frac{10}{6x-3}$.
6. $\frac{xy}{x^2-y^2}$. 7. $\frac{1}{x^2+5x+6}$.

Page 118. — 8. $\frac{1}{4x^2-1}$. 9. $\frac{2-4a}{a^2-a-12}$. 10. 1.

Page 120. — 1. $a+x$. 2. $\frac{a}{a+x}$. 3. a^2+a-2. 4. $\frac{x}{x+4}$.
5. $\frac{(x+y)^2}{x}$. 6. $\frac{3a}{2(a+b)}$. 7. $\frac{(1+x)^2}{1+x^2}$. 8. $\frac{x^2-x-20}{5}$.
9. $\frac{xy}{x+y}$. 10. $\frac{x}{y}$. 11. $\frac{c}{1+x}$. 12. $\frac{c}{b}$. 13. $\frac{4(x+2)}{3y}$. 14. $\frac{x-z}{3ax}$.
1. $\frac{a^4-b^4}{a^2b}$. 2. $\frac{3ay-6by-2a^2+4ab}{3by}$. 3. $\frac{a^{m+n}}{b^{m+n}}$.
4. $\frac{y-xy+x-1}{x}$. 5. $\frac{x+1}{x-1}$. 6. $\frac{a^2b(a-b)}{a^2-ab+b^2}$.

Page 122. — 1. -3. 2. 3. 3. 5. 4. 8. 5. 6.
1. 20. 2. 8. 3. $a+1-\frac{a}{b}$. 4. $\frac{1}{2a-1}$. 5. $\frac{an}{bm}$. 6. 2.

Page 125. — 1. 240. 2. 3. 3. 63. 4. 45. 5. $\frac{bc}{d}$. 6. 22.
7. 26. 8. 24. 9. 9. 10. 6.
1. $2a^2c^2$. 2. $\frac{1}{8}$.

Page 126. — 2. \$29. 3. 28. 4. 15. 5. 150 bu. 6. 28 men.
7. \$300. 8. 18 da. 9. 13 yr. 10. \$375. 11. 22 da.

Page 129. — **1.** $x = 2$, $y = 5$. **2.** $x = 3$, $y = 6$. **3.** $x = 1$, $y = 4$. **4.** $x = 4$, $y = 9$. **5.** $x = 2$, $y = 10$. **6.** $x = 5$, $y = 7$. **7.** $x = 5$, $y = 8$. **8.** $x = 9$, $y = 11$.

Page 130. — **9.** $x = 6$, $y = 5$. **10.** $x = 9$, $y = 12$. **11.** $x = 6$, $y = 1$. **12.** $x = 2$, $y = 9$. **13.** $x = -2$, $y = 4$. **14.** $y = 9$, $z = 4$. **15.** $x = 9$, $z = 5$. **16.** $x = 25$, $y = -10$.

Page 131. — **17.** $x = 6$, $y = 8$. **18.** $x = 24$, $y = 24$. **19.** $x = 36$, $y = 24$. **20.** $x = 18$, $y = 48$.

Page 132. — **1.** $x = 5$, $y = 10$. **2.** $x = 6$, $y = 8$. **3.** $x = 4$, $y = 7$. **4.** $x = 12$, $y = 8$. **5.** $x = 12$, $y = 17$. **6.** $x = 1$, $y = 1$.

Page 133. — **1.** $x=4$, $y=-2$. **2.** $x=\frac{1}{2}$, $y=4$. **3.** $x=40$, $y=24$. **4.** $x=8$, $y=9$.

1. $x = 4$, $y = 6$. **2.** $x = 6$, $y = 9$. **3.** $x = 7$, $y = 9$. **4.** $x = 4$, $y = 5$.

Page 134. — **5.** $x = 3$, $y = 10$. **6.** $x = 2$, $y = 12$.

1. $x = 3$, $y = 14$. **2.** $x = 13$, $y = 24$.

1. $x = 2$, $y = 10$. **2.** $x = 7$, $y = 21$. **3.** $x = 20$, $y = 5$. **4.** $x = 12$, $y = 18$. **5.** $x = 24$, $y = 6$. **6.** $x = 24$, $y = 12$. **7.** $x = 6$, $y = 8$. **8.** $x = 6$, $y = 14$.

Page 136. — **1.** Sheep, \$7; pigs, \$8. **2.** 12, 10. **3.** 45¢, 35¢. **4.** 44 yr., 40 yr. **5.** Men, \$2; boys, \$1. **6.** \$364, \$264. **7.** Father, 72 yr.; son, 48 yr. **8.** 120, 150.

Page 137. — **9.** $\frac{5}{8}$. **10.** $\frac{3}{4}$. **11.** $\frac{28}{29}$. **12.** A, \$30; B, \$40. **13.** 10 yr. and 40 yr. **14.** Carriage, \$150; horse, \$200. **15.** A, \$5000; B, \$7500. **16.** Man, 27 yr.; wife, 23 yr. **17.** 10 and 5.

Page 138. — **18.** First, 40¢; second, 30¢; third, 20¢; fourth, 10¢. **19.** A, \$2400; B, \$1200; C, \$1600; D, \$800. **20.** \$150, \$40. **21.** 30 and 20. **22.** 70 and 20. **23.** 9 in. and 8 in. **24.** 31 in. and 21 in.

Page 139. — **1.** \$2500; 20%. **2.** \$5000; 20%. **3.** \$400; 4 yr. **4.** \$2000; 8 yr. 6 mo. **5.** \$1600; 5%. **6.** \$2000; 7%.

Page 141. — **1.** ± 7. **2.** ± 9. **3.** ± 6. **4.** ± 5. **5.** ± 3. **6.** ± 4. **7.** ± 6. **8.** ± 5. **9.** ± 9. **10.** ± 2.

Page 142. — **11.** ± 6. **12.** ± 10. **13.** $\pm 2a$. **14.** ± 7. **15.** ± 3. **16.** ± 2.

2. 6. **3.** 3 and 9.

Page 143. — **4.** 4 and 8. **5.** 90 ft. and 50 ft. **6.** 30 ft. and 90 ft. **7.** 27. **8.** 16 and 4. **9.** 3 and 5. **10.** 7 and 11. **11.** 2 and 8. **12.** 4 ft. **13.** 4 and 8.

Page 144. — **1.** 6 and 10. **2.** 20. **3.** 4. **4.** 8 yr. **5.** 9.

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